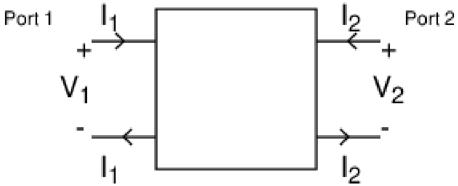
PH 207
ABCD Matrix

A. Mohanty

Department of Instrumentation and Applied Physics
Indian Institute of Science
Bangalore 560012

February 15, 2024

Impedance (Z) and Admittance (Y) Matrices



Let $V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$, and $I = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$.

Then $V = ZI$, and $I = YV$.

$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$ is called the **impedance** matrix.

$Y = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}$ is called the **admittance** matrix.

Properties of Z and Y Matrices

Of course, $Y = Z^{-1}$, and $Z = Y^{-1}$.
Advantages of Z and Y:

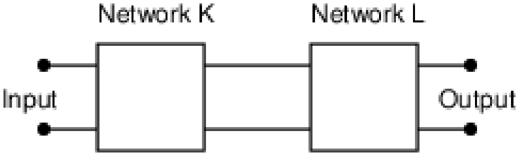
- Provide a simple description.
- Can be generalized to n-port networks.

Disadvantages of Z and Y:

- Do not help for cascaded connection of two-port networks.
- Not easy to see how the load impedance gets transformed.

The Cascading of Two-port Networks

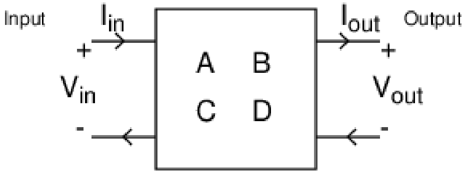
This is the most common way of combining two netwoks.



- Given Z_K and Z_L, how do we find Z of the cascaded network?
- Given Y_K and Y_L, how do we find Y of the cascaded network?
- No easy answer.

The transmission matrix, or the ABCD matrix description provides the simplest formula for a cascaded network.

The ABCD Matrix

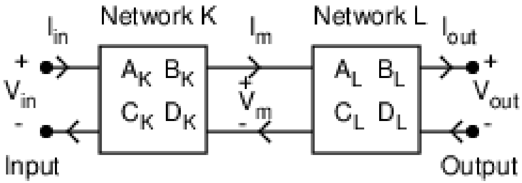


$$\begin{bmatrix} V_{\text{in}} \\ I_{\text{in}} \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \begin{bmatrix} V_{\text{out}} \\ I_{\text{out}} \end{bmatrix}$$

Points to note:

- Input V and I are given in terms of output V and I.
- The output current flows out of the block.
- $T = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}$ is called the transmission matrix.

The ABCD Matrix of a Cascade Connection



$$\begin{bmatrix} V_{\mathrm{in}} \\ I_{\mathrm{in}} \end{bmatrix} = \begin{bmatrix} \mathcal{A}_K & \mathcal{B}_K \\ \mathcal{C}_K & \mathcal{D}_K \end{bmatrix} \begin{bmatrix} V_{\mathrm{m}} \\ I_{\mathrm{m}} \end{bmatrix} = \begin{bmatrix} \mathcal{A}_K & \mathcal{B}_K \\ \mathcal{C}_K & \mathcal{D}_K \end{bmatrix} \begin{bmatrix} \mathcal{A}_L & \mathcal{B}_L \\ \mathcal{C}_L & \mathcal{D}_L \end{bmatrix} \begin{bmatrix} V_{\mathrm{out}} \\ I_{\mathrm{out}} \end{bmatrix}$$

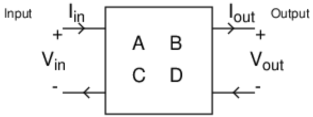
So the ABCD matrix of the combined network is

$$\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} = \begin{bmatrix} \mathcal{A}_K & \mathcal{B}_K \\ \mathcal{C}_K & \mathcal{D}_K \end{bmatrix} \begin{bmatrix} \mathcal{A}_L & \mathcal{B}_L \\ \mathcal{C}_L & \mathcal{D}_L \end{bmatrix}$$

which is nothing but the product of the ABCD matrices of the component networks from left to right.

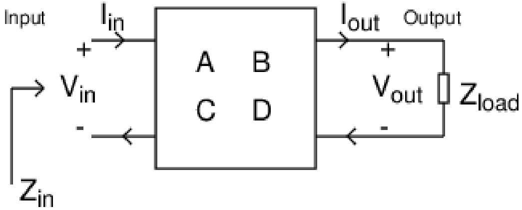
This makes the ABCD matrix very useful in studying practical networks made by cascading simpler networks.

ABCD Matrix Elements



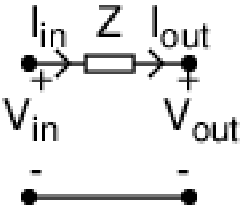
$V_{\text{in}} = \mathcal{A} V_{\text{out}} + \mathcal{B} I_{\text{out}}$
$I_{\text{in}} = \mathcal{C} V_{\text{out}} + \mathcal{D} I_{\text{out}}$
Measurement definitions:
$\mathcal{A} = \left. \frac{V_{\text{in}}}{V_{\text{out}}} \right|_{I_{\text{out}}=0}$, and $\mathcal{B} = \left. \frac{V_{\text{in}}}{I_{\text{out}}} \right|_{V_{\text{out}}=0}$.
$\mathcal{C} = \left. \frac{I_{\text{in}}}{V_{\text{out}}} \right|_{I_{\text{out}}=0}$, and $\mathcal{D} = \left. \frac{I_{\text{in}}}{I_{\text{out}}} \right|_{V_{\text{out}}=0}$.
Dimensions: $\mathcal{A}$ and $\mathcal{D}$ are dimensionless. $\mathcal{B}$ is an impedance. $\mathcal{C}$ is an admittance.
Note that $\mathcal{A}$ and $\mathcal{C}$ are measured with the output open circuited, while $\mathcal{B}$ and $\mathcal{D}$ are measured with the output short circuited.
Note: The open circuit transfer function $T(s) = \frac{1}{\mathcal{A}}$.

Impedance Transformation



$Z_{\text{in}} = \frac{V_{\text{in}}}{I_{\text{in}}} = \frac{\mathcal{A}V_{\text{out}}+\mathcal{B}I_{\text{out}}}{\mathcal{C}V_{\text{out}}+\mathcal{D}I_{\text{out}}} = \frac{\mathcal{A}V_{\text{out}}/I_{\text{out}}+\mathcal{B}}{\mathcal{C}V_{\text{out}}/I_{\text{out}}+\mathcal{D}} = \frac{\mathcal{A}Z_{\text{load}}+\mathcal{B}}{\mathcal{C}Z_{\text{load}}+\mathcal{D}}$
since $V_{\text{out}}/I_{\text{out}} = Z_{\text{load}}$.
Möbius transformation or linear fractional transformation.
Where else do you see such transformations?

Series Element



Note: The element **must** be written as an **impedance.**

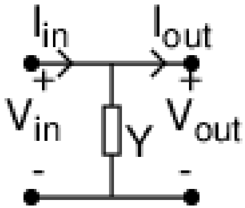
$V_{\text{in}} = V_{\text{out}} + ZI_{\text{out}}$

$I_{\text{in}} = I_{\text{out}} = 0\,V_{\text{out}} + I_{\text{out}}$

$\Rightarrow \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} = \begin{bmatrix} 1 & Z \\ 0 & 1 \end{bmatrix}$

What is the determinant of this matrix?

Shunt Element



Note: The element **must** be written as an **admittance.**

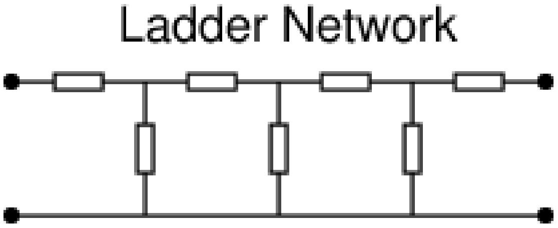
$V_{\text{in}} = V_{\text{out}} = V_{\text{out}} + 0I_{\text{out}}$

$I_{\text{in}} = YV_{\text{out}} + I_{\text{out}}$

$\Rightarrow \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ Y & 1 \end{bmatrix}$

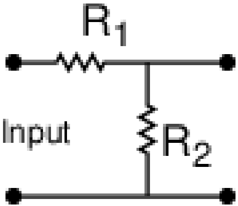
What is the determinant of this matrix?

Ladder Network as a Cascade



A ladder network can be considered as a cascade of series and shunt elements.

The Voltage Divider



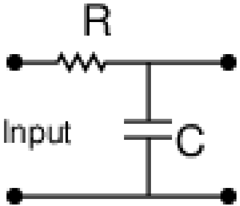
The ABCD matrix of this network is

$$\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} = \begin{bmatrix} 1 & R_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1/R_2 & 1 \end{bmatrix} = \begin{bmatrix} 1 + R_1/R_2 & R_1 \\ 1/R_2 & 1 \end{bmatrix}$$

Verify that $T(s) = \frac{1}{\mathcal{A}} = \frac{R_2}{R_1+R_2}$.

Note that for the shunt resistor, the entry in the matrix was for the $\mathcal{C}$ element, and was converted to the admittance $1/R_2$ first.

The RC Lowpass Filter



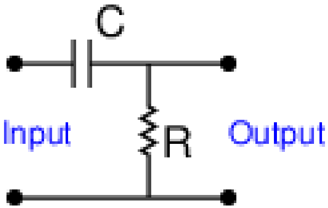
The ABCD matrix of this network is

$$\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} = \begin{bmatrix} 1 & R \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ sC & 1 \end{bmatrix} = \begin{bmatrix} 1 + sRC & R \\ sC & 1 \end{bmatrix}$$

Verify that $T(s) = \frac{1}{\mathcal{A}} = \frac{1}{1+sRC} = \frac{\frac{1}{RC}}{s+\frac{1}{RC}} = \frac{\omega_0}{s+\omega_0}$,

where $\omega_0 = \frac{1}{RC}$.

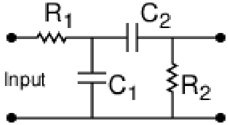
The CR Highpass Filter



The ABCD matrix of this network is
$$\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{sC} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{1}{R} & 1 \end{bmatrix} = \begin{bmatrix} 1 + \frac{1}{sRC} & \frac{1}{sC} \\ \frac{1}{R} & 1 \end{bmatrix}$$
Verify that $T(s) = \frac{1}{\mathcal{A}} = \frac{1}{1+\frac{1}{sRC}} = \frac{s}{s+\frac{1}{RC}} = \frac{s}{s+\omega_0}$,

where $\omega_0 = \frac{1}{RC}$.

A Bandpass Filter



The ABCD matrix of this network is

$$\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} = \begin{bmatrix} 1 & R_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ sC_1 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{sC_2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{1}{R_2} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + sR_1C_1 & R_1 \\ sC_1 & 1 \end{bmatrix} \begin{bmatrix} 1 + \frac{1}{sR_2C_2} & \frac{1}{sC_2} \\ \frac{1}{R_2} & 1 \end{bmatrix}$$

We only write down the $\mathcal{A}$ element of the resulting matrix.

$\mathcal{A} = sR_1C_1 + 1 + \frac{R_1}{R_2} + \frac{R_1C_1}{R_2C_2} + \frac{1}{sR_2C_2}$.

At what frequency is $T(s) = \frac{1}{\mathcal{A}}$ real?

Answer: $f_0 = \frac{1}{2\pi\sqrt{R_1R_2C_1C_2}}$

What is $T(s)$ at that frequency?

Answer: $1/(1 + \frac{R_1}{R_2} + \frac{R_1C_1}{R_2C_2})$.

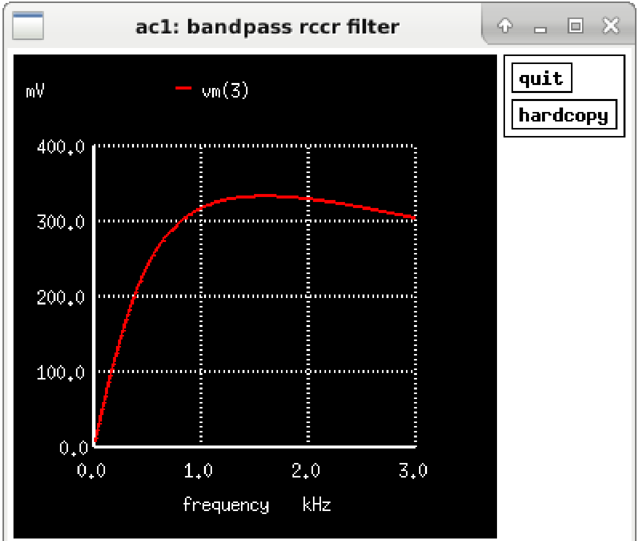
SPICE Code

File rccr.cir:

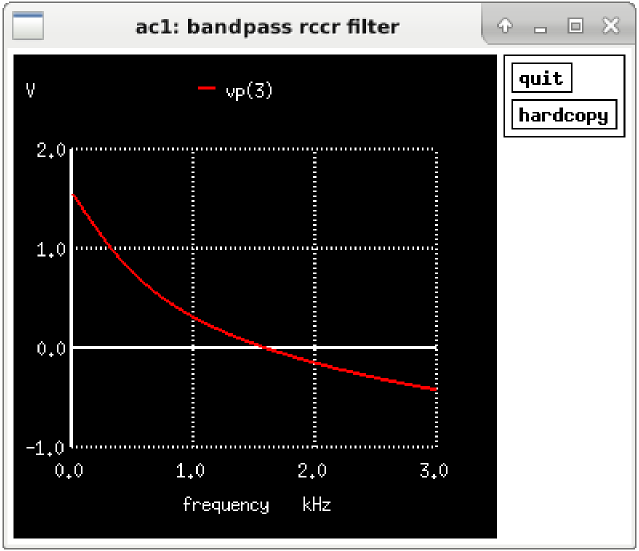
```
Bandpass RCCR Filter
*****************************
VIN 1 0 AC 1
R1 1 2 10k
C1 2 0 10n
C2 2 3 10n
R2 3 0 10k
.AC LIN 1000 10 3k

.control
run
plot vm(3)
plot vp(3)
.endcontrol
.END
```

SPICE Results: Magnitude Plot

On Linux, you can type
ngspice rccr.cir



SPICE Results: Phase Plot



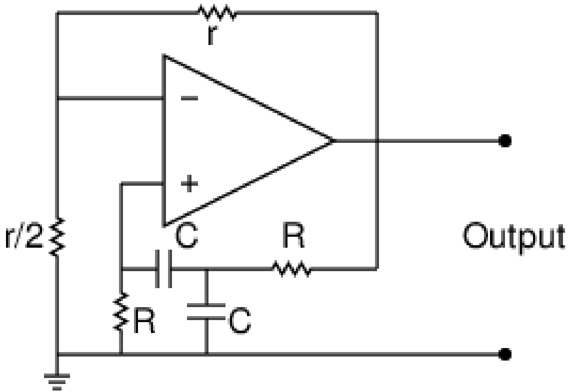
Available SPICE Software

- ngspice for Linux and OpenBSD (Recommended)
- LTspice for Windows

Making a Sinewave Oscillator

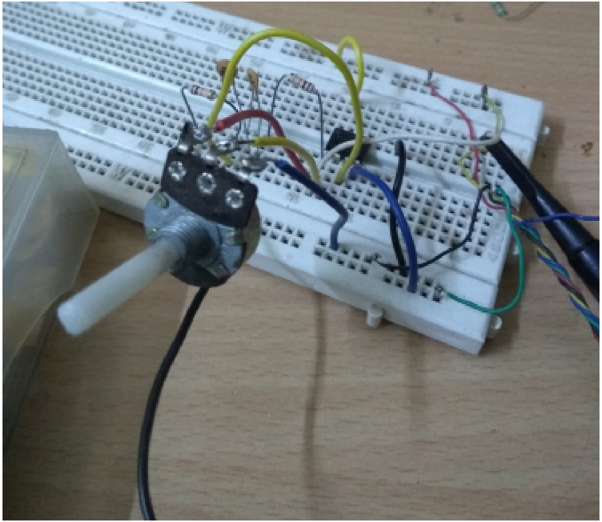
Let $R_1 = R_2 = R$, and $C_1 = C_2 = C$ in the circuit discussed.
Then $f_0 = \frac{1}{2\pi RC}$.
If $R = 10\,\mathrm{k\Omega}$, and $C = 10\,\mathrm{nF}$, $f_0 = 1.591\,55\,\mathrm{kHz}$.
$T(s)$ at this frequency is $1/3$.
So if we make a voltage amplifier of gain $+3$, we may be able to make a sinewave oscillator if we use this circuit in the feedback path.

Circuit Diagram: No AGC

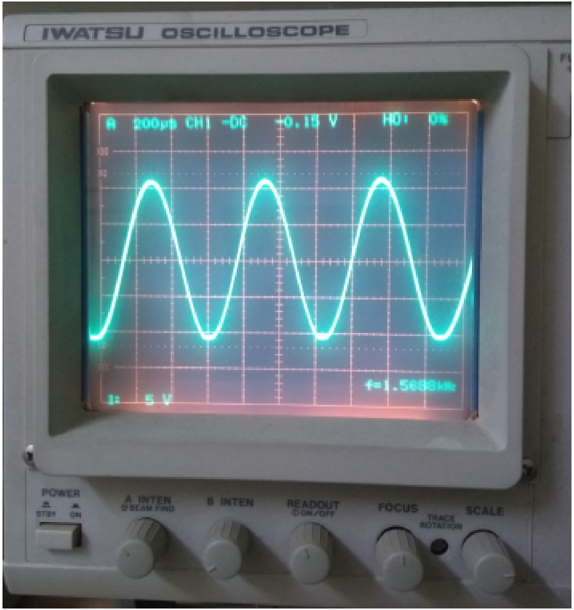


Will either fail to oscillate or give clipped output.

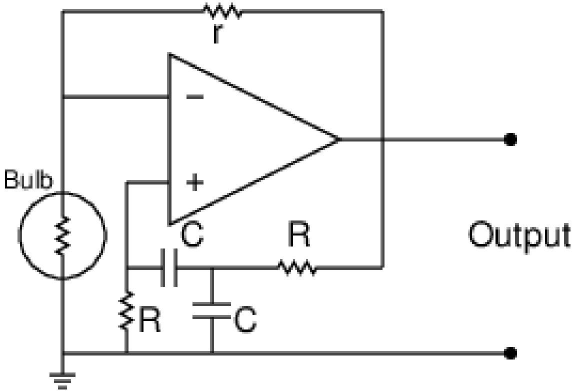
Bad Circuit



Bad Output: Clipped output

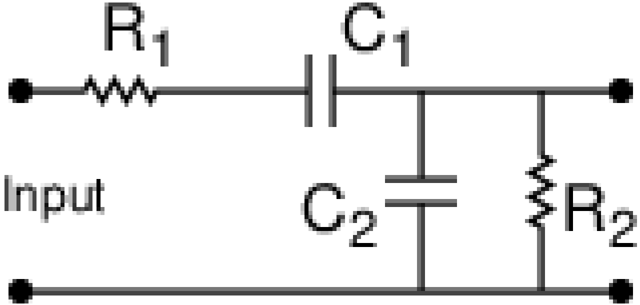


Circuit Diagram: With AGC



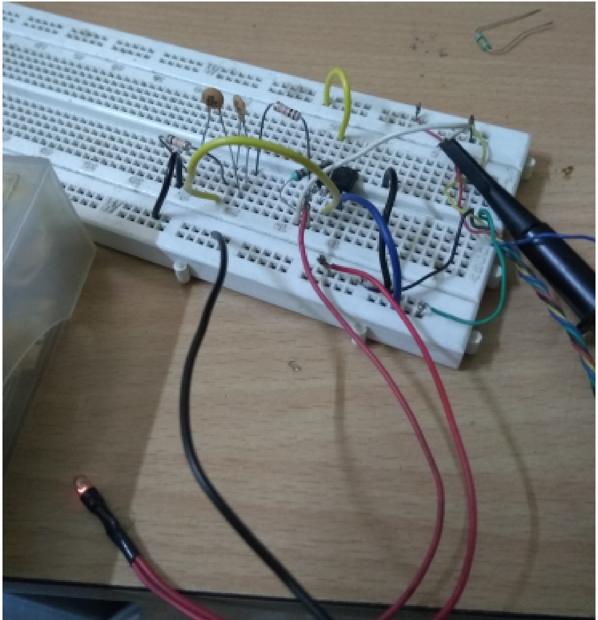
Can be made to work very well.
The success of Hewlett-Packard HP200A!
Note: HP200A uses a Wien bridge circuit which is slightly different.

Wien Bridge Circuit

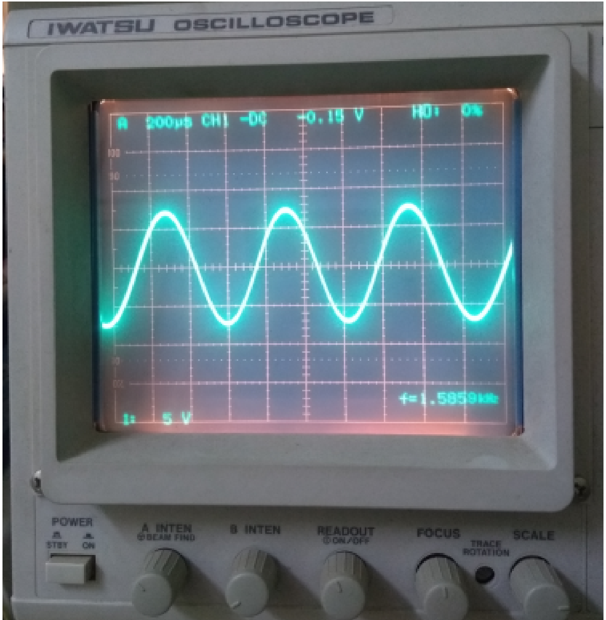


Note: Not used in our circuit.

Good Circuit



Good Output: No clipping



Summary

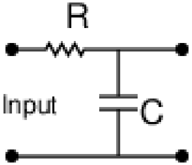
The ABCD matrix . . .

- . . . simplifies circuit analysis.
- . . . will often be used in this course.

Simple RLC Circuits as Building Blocks

1. First order RC or RL circuits.
2. Second order RLC circuits.
3. Second order RC circuits. (To be discussed in the next class.)

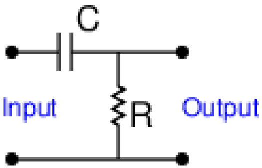
RC LPF



$$T(s) = \frac{\omega_0}{s + \omega_0}$$

where, $\omega_0 = \frac{1}{RC}$

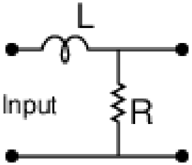
CR HPF



$$T(s) = \frac{s}{s + \omega_0}$$

where, $\omega_0 = \frac{1}{RC}$

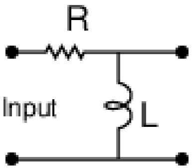
LR LPF



$$T(s) = \frac{\omega_0}{s + \omega_0}$$

where, $\omega_0 = \frac{R}{L}$

RL HPF



$$T(s) = \frac{s}{s + \omega_0}$$

where, $\omega_0 = \frac{R}{L}$

First Order LPF Transfer Function

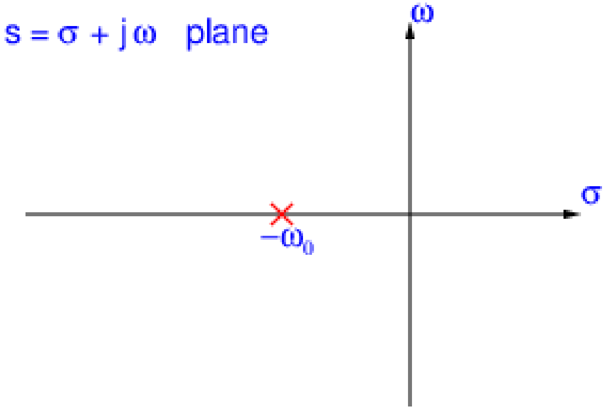
$$T(s) = \frac{\omega_0}{s + \omega_0}$$

$$T(j\omega) = \frac{1}{1 + j\omega/\omega_0}$$

$$|T(j\omega)| = \frac{1}{\sqrt{1 + (\omega/\omega_0)^2}}$$

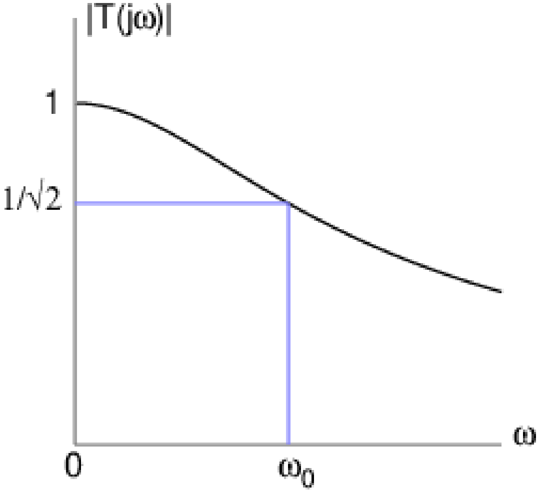
So $|T(j\omega_0)| = 1/\sqrt{2}$.
For $|\omega| \gg \omega_0$, $|T(j\omega)| \approx \omega_0/|\omega|$.

First Order LPF Pole-zero Diagram

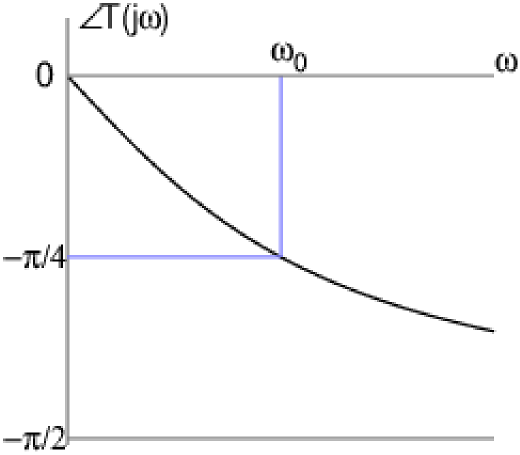


Has one pole and no zero.

First Order LPF TF Magnitude Plot



First Order LPF TF Phase Plot



First Order HPF Transfer Function

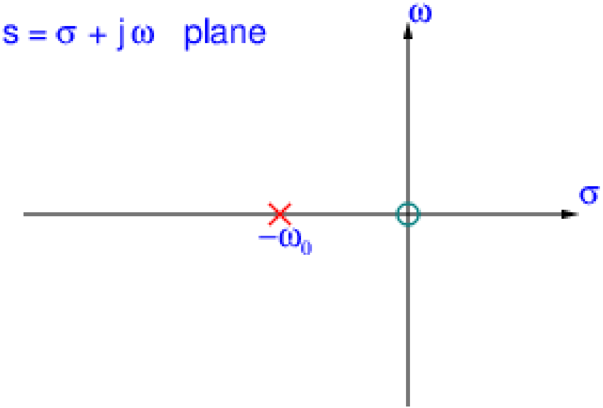
$$T(s) = \frac{s}{s + \omega_0}$$

$$T(j\omega) = \frac{1}{1 - j\omega_0/\omega}$$

$$|T(j\omega)| = \frac{1}{\sqrt{1 + (\omega_0/\omega)^2}}$$

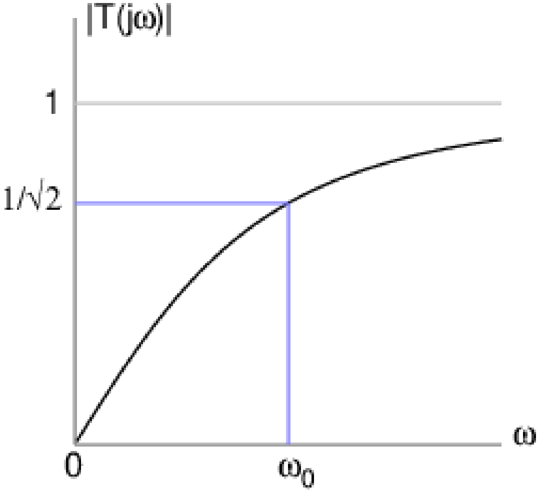
So $|T(j\omega_0)| = 1/\sqrt{2}$.
For $|\omega| \ll \omega_0$, $|T(j\omega)| \approx |\omega|/\omega_0$.

First Order HPF Pole-zero Diagram

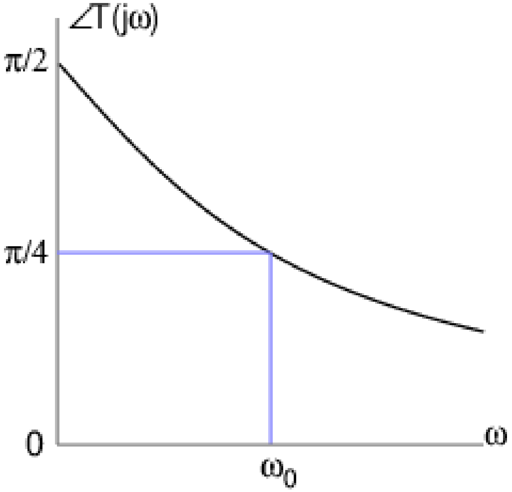


Has one pole and one zero.

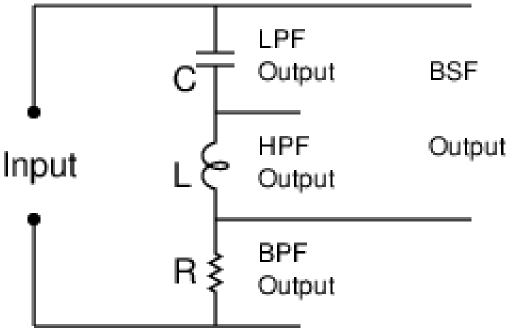
First Order HPF TF Magnitude Plot



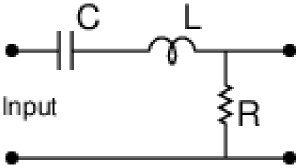
First Order HPF TF Phase Plot



The Series RLC Circuit



The Series RLC Bandpass Filter



$$T(s) = \frac{R}{sL + R + \frac{1}{sC}}$$

Simplify to get

$$T(s) = \frac{\frac{R}{L}s}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Second-order Transfer Functions: LPF, BPF, and HPF

Now we recall the second-order transfer functions connected with the spring-mass-dashpot system.
LPF (Lowpass Filter):

$$T_{\mathrm{LPF}}(s) = \frac{\omega_0^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}. \tag{1}$$

BPF (Bandpass Filter):

$$T_{\mathrm{BPF}}(s) = \frac{\frac{\omega_0}{Q}s}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}. \tag{2}$$

HPF (Highpass Filter):

$$T_{\mathrm{HPF}}(s) = \frac{s^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}. \tag{3}$$

When discussing a particular type of filter, the subscript of T may be omitted.

The Second-order Bandpass Transfer Function

$$T(s) = \frac{2\alpha s}{s^2 + 2\alpha s + \omega_0^2}$$

For small loss, that is for small b, or for small α, $T(s)$ has poles at $-\alpha \pm j\sqrt{\omega_0^2 - \alpha^2}$.
So α is the decay constant.
ω_0 is the angular frequency of oscillations for no loss.

Magnitude Response in the Frequency Domain

$$T(s) = \frac{2\alpha s}{s^2 + 2\alpha s + \omega_0^2}$$

$$T(j\omega) = \frac{j2\alpha\omega}{-\omega^2 + j2\alpha\omega + \omega_0^2} = \frac{1}{1 + \frac{\omega_0^2 - \omega^2}{j2\alpha\omega}} = \frac{1}{1 + j\frac{\omega^2 - \omega_0^2}{2\alpha\omega}}$$

Centre Angular Frequency

$$T(j\omega) = \frac{1}{1 + j\frac{\omega^2 - \omega_0^2}{2\alpha\omega}}$$

When is $|T(j\omega)| = 1$?
This happens when $\omega = \pm\omega_0$.
At other values of ω, $|T(j\omega)| < 1$.

Half-power Angular Frequencies

$$T(j\omega) = \frac{1}{1 + j\frac{\omega^2 - \omega_0^2}{2\alpha\omega}}$$

When is $|T(j\omega)| = \frac{1}{\sqrt{2}}$?

This happens when $\frac{\omega^2 - \omega_0^2}{2\alpha\omega} = \pm 1$.

Or, $\omega^2 - \omega_0^2 = \pm 2\alpha\omega$.

The two quadratic equations are,

$\omega^2 - 2\alpha\omega - \omega_0^2 = 0$,

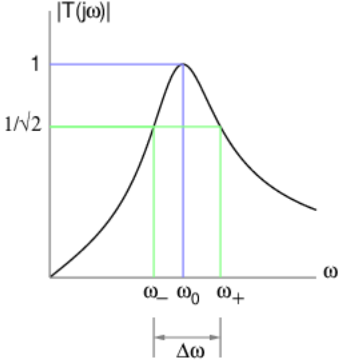
and

$\omega^2 + 2\alpha\omega - \omega_0^2 = 0$.

The positive root of the first quadratic equation is $\omega_+ = \alpha + \sqrt{\alpha^2 + \omega_0^2}$.

The positive root of the second quadratic equation is $\omega_- = -\alpha + \sqrt{\alpha^2 + \omega_0^2}$.

Magnitude Plot of the BPF Transfer Function



Note that $\omega_+\omega_- = \omega_0^2$. Half-power angular bandwidth: $\Delta\omega = \omega_+ - \omega_- = 2\alpha$.
Quality factor

$$Q = \frac{\omega_0}{\Delta\omega} = \frac{\omega_0}{2\alpha}$$

What is Q?

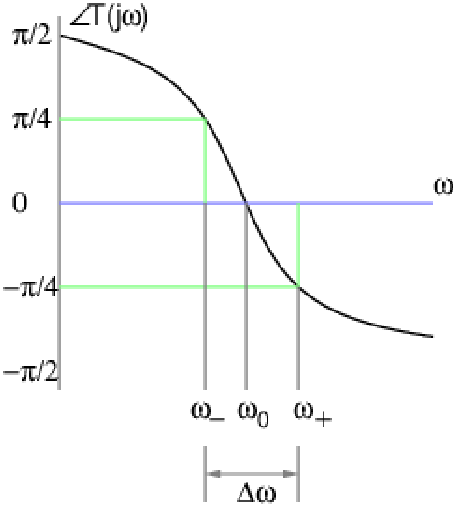
Q is a measure of the selectivity of the BPF. Note that this definition in the frequency domain is the original, exact definition of Q.
Note that $2\alpha = \Delta\omega = \frac{\omega_0}{Q}$.

$$\omega_+ = \left(\sqrt{1 + \frac{1}{4Q^2}} + \frac{1}{2Q}\right)\omega_0$$

$$\omega_- = \left(\sqrt{1 + \frac{1}{4Q^2}} - \frac{1}{2Q}\right)\omega_0$$

Remember that ω_0 is the *geometric* mean of ω_+ and ω_-.
It is NOT the arithmetic mean of ω_+ and ω_-.

Phase Plot of the BPF Transfer Function



Phase is easier to measure!

BPF Transfer Function Rewritten

$$T(s) = \frac{2\alpha s}{s^2 + 2\alpha s + \omega_0^2}$$

Since $2\alpha = \frac{\omega_0}{Q}$,

$$T(s) = \frac{\frac{\omega_0}{Q} s}{s^2 + \frac{\omega_0}{Q} s + \omega_0^2}$$

This is the standard form of the transfer function of the BPF.
For the mass-spring-dashpot BPF system, $\omega_0 = \sqrt{k/m}$, and $2\alpha = b/m$. So,

$$Q = \frac{\omega_0}{2\alpha} = \frac{\sqrt{k/m}}{b/m} = \frac{\sqrt{km}}{b}. \tag{4}$$

For other circuits or physical systems, these expressions will need to be determined in terms of the parameters of that system.

The Second Order Bandpass Transfer Function

$$T(s) = \frac{\frac{R}{L}s}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Write $\frac{1}{LC} = \omega_0^2$, and $\frac{R}{L} = 2\alpha$ to get

$$T(s) = \frac{2\alpha s}{s^2 + 2\alpha s + \omega_0^2}$$

For small loss, that is for small R, or for small α, $T(s)$ has poles at $-\alpha \pm j\sqrt{\omega_0^2 - \alpha^2}$.
So α is the decay constant.
ω_0 is the angular frequency of oscillations for no loss.

Magnitude Response in the Frequency Domain

$$T(s) = \frac{2\alpha s}{s^2 + 2\alpha s + \omega_0^2}$$

$$T(j\omega) = \frac{j2\alpha\omega}{-\omega^2 + j2\alpha\omega + \omega_0^2} = \frac{1}{1 + \frac{\omega_0^2 - \omega^2}{j2\alpha\omega}} = \frac{1}{1 + j\frac{\omega^2 - \omega_0^2}{2\alpha\omega}}$$

Centre Angular Frequency

$$T(j\omega) = \frac{1}{1 + j\frac{\omega^2 - \omega_0^2}{2\alpha\omega}}$$

When is $|T(j\omega)| = 1$?
This happens when $\omega = \pm\omega_0$.
At other values of ω, $|T(j\omega)| < 1$.

Half-power Angular Frequencies

$$T(j\omega) = \frac{1}{1 + j\frac{\omega^2 - \omega_0^2}{2\alpha\omega}}$$

When is $|T(j\omega)| = \frac{1}{\sqrt{2}}$?

This happens when $\frac{\omega^2 - \omega_0^2}{2\alpha\omega} = \pm 1$.

Or, $\omega^2 - \omega_0^2 = \pm 2\alpha\omega$.

The two quadratic equations are,

$\omega^2 - 2\alpha\omega - \omega_0^2 = 0$,

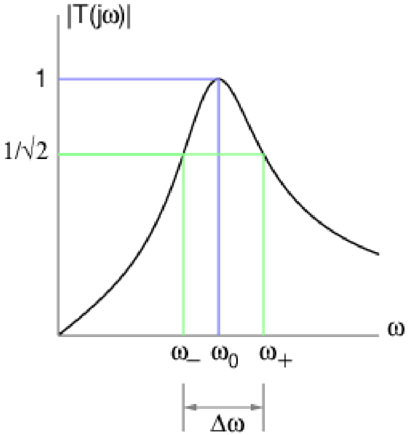
and

$\omega^2 + 2\alpha\omega - \omega_0^2 = 0$.

The positive root of the first quadratic equation is $\omega_+ = \alpha + \sqrt{\alpha^2 + \omega_0^2}$.

The positive root of the second quadratic equation is $\omega_- = -\alpha + \sqrt{\alpha^2 + \omega_0^2}$.

Magnitude Plot of the BPF Transfer Function



Note that $\omega_+\omega_- = \omega_0^2$.
Half-power angular bandwidth: $\Delta\omega = \omega_+ - \omega_- = 2\alpha$.
Quality factor

$\omega_0 \qquad \omega_0$

What is Q?

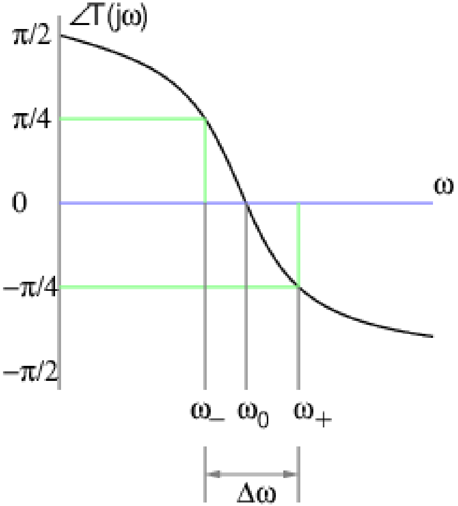
Q is a measure of the selectivity of the BPF. Note that this definition in the frequency domain is the original, exact definition of Q.
Note that $2\alpha = \Delta\omega = \frac{\omega_0}{Q}$.

$$\omega_+ = \left(\sqrt{1 + \frac{1}{4Q^2}} + \frac{1}{2Q}\right)\omega_0$$

$$\omega_- = \left(\sqrt{1 + \frac{1}{4Q^2}} - \frac{1}{2Q}\right)\omega_0$$

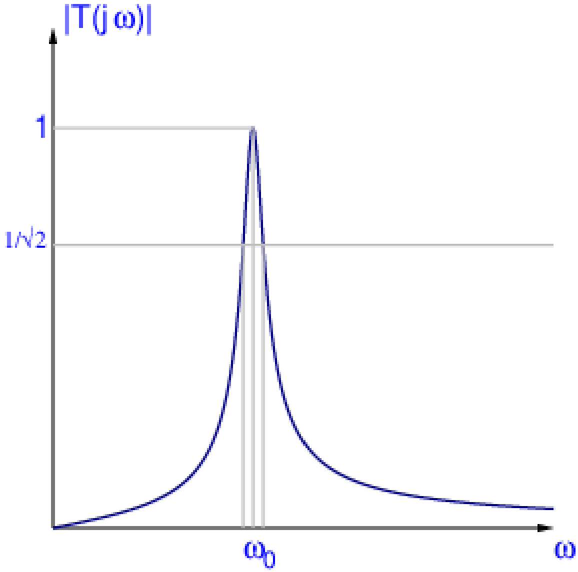
Remember that ω_0 is the *geometric* mean of ω_+ and ω_-.
It is NOT the arithmetic mean of ω_+ and ω_-.

Phase Plot of the BPF Transfer Function

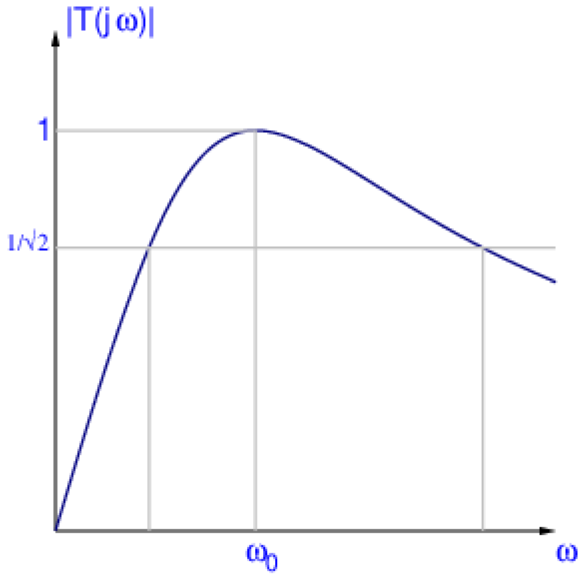


Phase is easier to measure!

$|T(j\omega)|$ for $Q = 10$



$|T(j\omega)|$ for $Q = 0.6$



BPF Phase

$$T(j\omega) = \frac{2\alpha j\omega}{2\alpha j\omega + \omega_0^2 - \omega^2} = \frac{j\omega\omega_0/Q}{j\omega\omega_0/Q + \omega_0^2 - \omega^2} = \frac{1}{1 + jQ\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)}. \tag{5}$$

Phase angle is

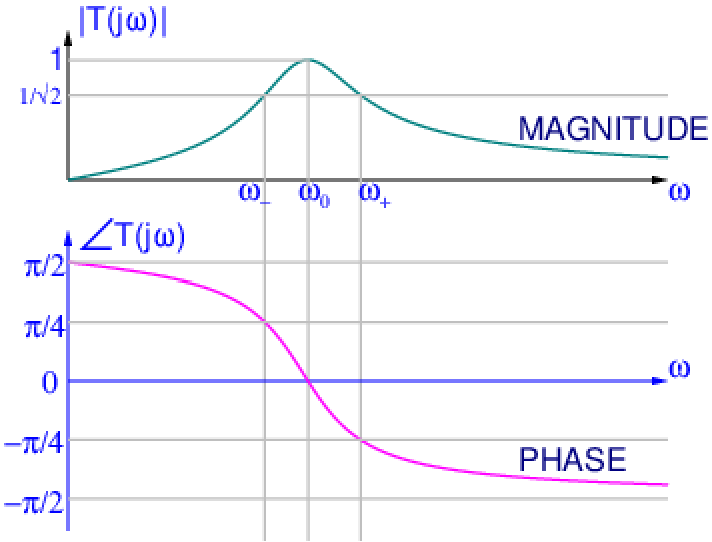
$$\underline{/T(j\omega)} = \text{arctan}\left(Q\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)\right). \tag{6}$$

Special values:

- $\underline{/T(j0)} = \pi/2$.
- $\underline{/T(j\omega_0)} = 0$.
- $\underline{/T(j\infty)} = -\pi/2$.
- $\underline{/T(j\omega_-)} = \pi/4$.
- $\underline{/T(j\omega_+)} = -\pi/4$.

Phase is important because it is often easier to measure.

BPF magnitude and phase on the same plot



BPF MAGNITUDE AND PHASE PLOTS FOR Q = 2.5

BPF Transfer Function Rewritten

$$T(s) = \frac{2\alpha s}{s^2 + 2\alpha s + \omega_0^2}$$

Since $2\alpha = \frac{\omega_0}{Q}$,

$$T(s) = \frac{\frac{\omega_0}{Q} s}{s^2 + \frac{\omega_0}{Q} s + \omega_0^2}$$

This is the standard form of the transfer function of the BPF.
For the series RLC BPF,

$$\omega_0 = 1/\sqrt{LC},$$

and

$$Q = \frac{\omega_0}{2\alpha} = \frac{\frac{1}{\sqrt{LC}}}{\frac{R}{L}} = \frac{\sqrt{L/C}}{R}.$$

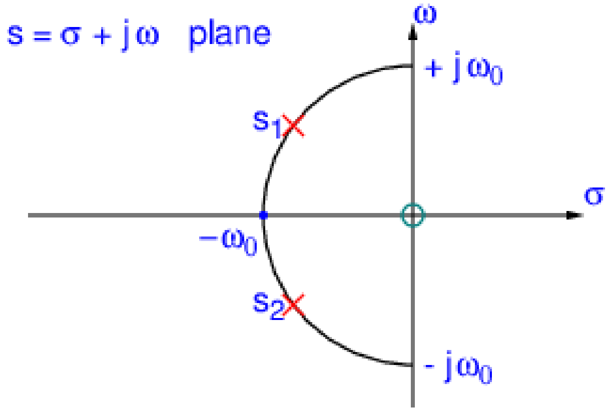
For other circuits or physical systems, these expressions will need to be determined in terms of the parameters of that system.

General Second Order BPF Transfer Function

$$T(s) = \frac{H\frac{\omega_0}{Q}s}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$$

ω_0: Centre angular frequency
Q: Quality factor
H: Gain factor

Second Order BPF Pole-zero Diagram



Shown for $Q > \frac{1}{2}$. Has two poles and one zero.

Second Order BPF Pole Locations

Find zeros of $s^2 + \frac{\omega_0}{Q}s + \omega_0^2$.

Case $Q > \frac{1}{2}$ (Underdamped)

$$s_1 = -\frac{\omega_0}{2Q} + j\omega_0\sqrt{1 - \frac{1}{4Q^2}}$$

$$s_2 = -\frac{\omega_0}{2Q} - j\omega_0\sqrt{1 - \frac{1}{4Q^2}}$$

Complex conjugate pair of poles. $s_1 s_2 = \omega_0^2$.

Case $Q = \frac{1}{2}$ (Critically damped)

$$s_1 = s_2 = -\omega_0.$$

Equal, negative real poles.

Second Order BPF Pole Locations

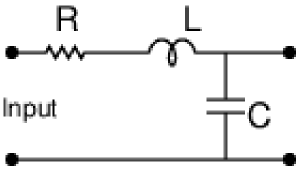
Case $Q < \frac{1}{2}$ (Overdamped)

$$s_1 = -\frac{\omega_0}{2Q} + \omega_0\sqrt{\frac{1}{4Q^2} - 1}$$

$$s_2 = -\frac{\omega_0}{2Q} - \omega_0\sqrt{\frac{1}{4Q^2} - 1}$$

Unequal negative real poles. $s_1 s_2 = \omega_0^2$.

The Series RLC Lowpass Filter



$$T(s) = \frac{\frac{1}{sC}}{sL + R + \frac{1}{sC}}$$

Simplify to get

$$T(s) = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} = \frac{\omega_0^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$$

Second Order LPF Magnitude Response

$$T(j\omega) = \frac{\omega_0^2}{\omega_0^2 - \omega^2 + j\frac{\omega\omega_0}{Q}}$$

At what frequency is $|T(j\omega)|$ maximum?
The numerator is constant. The square of the magnitude of the denominator is

$$(\omega_0^2 - \omega^2)^2 + \frac{\omega^2\omega_0^2}{Q^2} = \omega_0^4 + \omega^4 - 2\omega_0^2\omega^2 + \frac{\omega^2\omega_0^2}{Q^2}$$

$$= \omega_0^4 + \omega^4 - 2\omega_0^2\omega^2\left(1 - \frac{1}{2Q^2}\right)$$

We will try to complete squares here. The result depends on the value of Q.

Magnitude Response (continued)

If $Q \leq 1/\sqrt{2}$, all terms are non-negative and the denominator is an increasing function of ω.
In that case, $|T(j\omega)|$ has a maximum value of 1 at $\omega = 0$. For any other ω, $|T(j\omega)|$ is a monotonically decreasing function of $|\omega|$. We then say that there is *no peaking*.

Magnitude Response (continued)

If $Q > 1/\sqrt{2}$, we can complete the square to get the denominator magnitude squared as

$$\left(\omega^2 - \omega_0^2\left(1 - \frac{1}{2Q^2}\right)\right)^2 + \omega_0^4\frac{1}{Q^2}\left(1 - \frac{1}{4Q^2}\right)$$

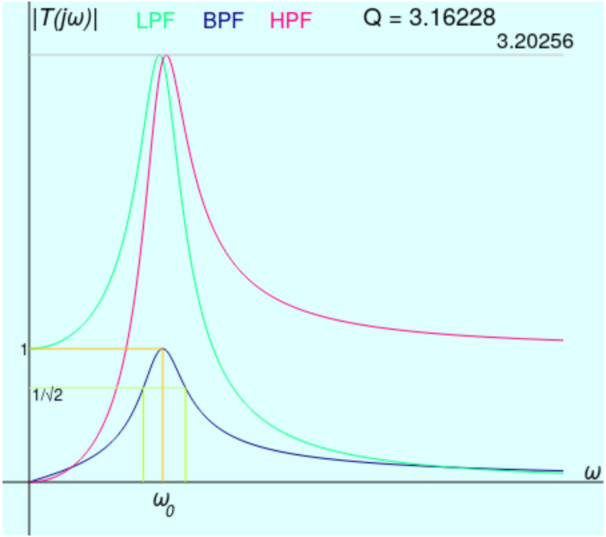
So $|T(j\omega)|$ is maximum when

$$|\omega| = \omega_L = \omega_0\sqrt{1 - \frac{1}{2Q^2}}.$$

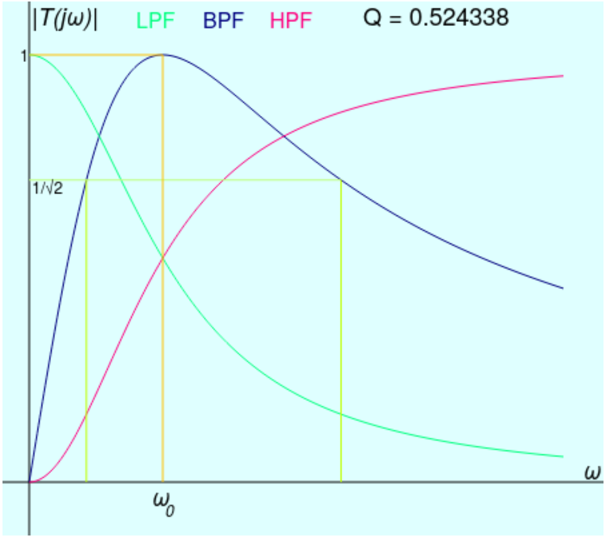
$$|T(j\omega_L)| = \frac{Q}{\sqrt{1 - \frac{1}{4Q^2}}}$$

This gives rise to *peaking*.

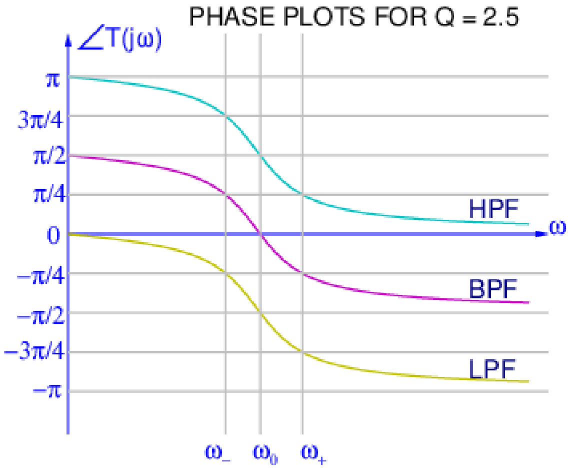
Case of *Peaking*



Case of *No Peaking*



$T(j\omega)$ Phase



Observations

Note that $T_{\text{HPF}}(j\omega)/T_{\text{BPF}}(j\omega) = jQ\omega/\omega_0$, and $T_{\text{LPF}}(j\omega)/T_{\text{BPF}}(j\omega) = -jQ\omega_0/\omega$.
So for positive ω, the HPF phase leads the BPF phase by $\pi/2$, while the LPF phase lags the BPF phase by $\pi/2$, as the plot shows.
In the same way, for the first-order case, HPF phase leads the LPF phase by $\pi/2$.
Points to note:

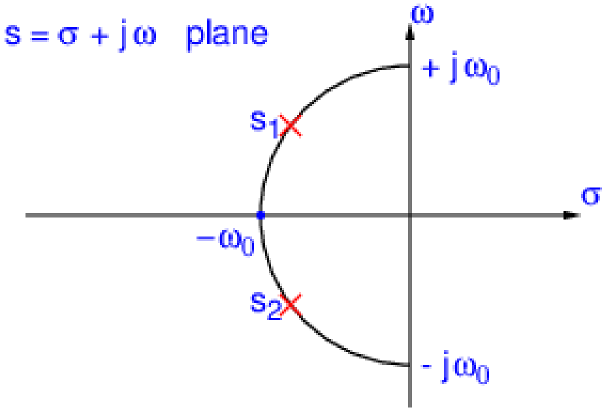
- Unlike the magnitude plots, the phase plots are monotonic.
- HPF, BPF, and LPF phase plots are very simply related to one another.
- Phase is often easier to measure.

General Second Order LPF Transfer Function

$$T(s) = \frac{H\omega_0^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$$

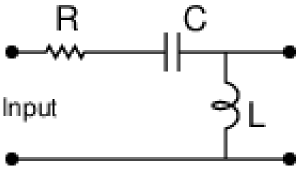
ω_0: Centre angular frequency
Q: Quality factor
H: Gain factor

Second Order LPF Pole-zero Diagram



Shown for $Q > \frac{1}{2}$. Has two poles and no zero.

The Series RLC Highpass Filter



$$T(s) = \frac{sL}{sL + R + \frac{1}{sC}}$$

Simplify to get

$$T(s) = \frac{s^2}{s^2 + \frac{R}{L}s + \frac{1}{LC}} = \frac{s^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$$

HPF Magnitude Response

If $Q > 1/\sqrt{2}$, show that $|T(j\omega)|$ is maximum when

$$|\omega| = \omega_H = \frac{\omega_0}{\sqrt{1 - \frac{1}{2Q^2}}}.$$

$$|T(j\omega_H)| = \frac{Q}{\sqrt{1 - \frac{1}{4Q^2}}}$$

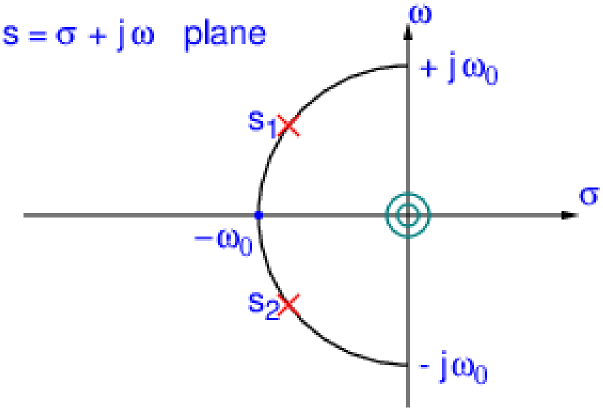
Note that $\omega_L \omega_H = \omega_0^2$, even though ω_H and ω_L refer to different types of filters.
If $Q \leq 1/\sqrt{2}$, then there is no peaking.

General Second Order HPF Transfer Function

$$T(s) = \frac{Hs^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$$

ω_0: Centre angular frequency
Q: Quality factor
H: Gain factor

Second Order HPF Pole-zero Diagram



Shown for $Q > \frac{1}{2}$. Has two poles and two zeros.

Notation and Terminology

Note that even though the second order LPF and HPF are not really bandpass filters, we still use the notations ω_0 and Q.
The meanings are different, even though the expressions are the same.

Broader Use of Q

Not all tuned systems are second order systems.
Still, the symbol Q is used in such systems.
One should be careful in such cases.