PH 207
ABCD Matrix

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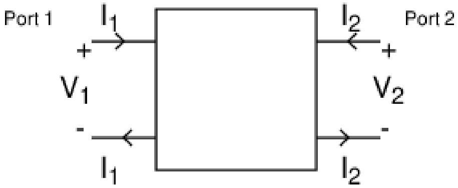
Impedance (Z) and Admittance (Y) Matrices



Let $V = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$, and $I = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$.

Then $V = ZI$, and $I = YV$.

$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$ is called the **impedance** matrix.

$Y = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix}$ is called the **admittance** matrix.

Properties of Z and Y Matrices

Of course, $Y = Z^{-1}$, and $Z = Y^{-1}$.
Advantages of Z and Y:

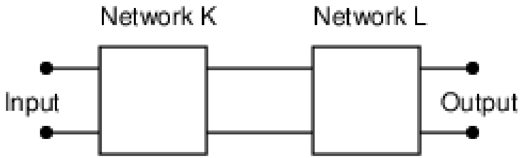
- Provide a simple description.
- Can be generalized to n-port networks.

Disadvantages of Z and Y:

- Do not help for cascaded connection of two-port networks.
- Not easy to see how the load impedance gets transformed.

The Cascading of Two-port Networks

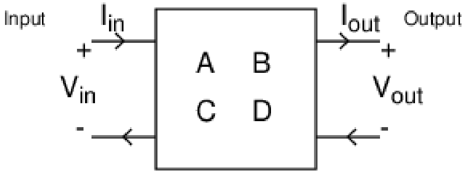
This is the most common way of combining two netwoks.



- Given Z_K and Z_L, how do we find Z of the cascaded network?
- Given Y_K and Y_L, how do we find Y of the cascaded network?
- No easy answer.

Another matrix, the transmission matrix, or the ABCD matrix description provides the simplest way to work with a cascaded network.

The ABCD Matrix

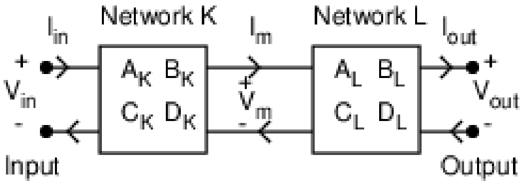


$$\begin{bmatrix} V_{\text{in}} \\ I_{\text{in}} \end{bmatrix} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \begin{bmatrix} V_{\text{out}} \\ I_{\text{out}} \end{bmatrix}$$

Points to note:

- Input V and I are given in terms of output V and I.
- The output current flows out of the block.
- $T = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}$ is called the transmission matrix.

The ABCD Matrix of a Cascade Connection



$$\begin{bmatrix} V_{\text{in}} \\ I_{\text{in}} \end{bmatrix} = \begin{bmatrix} \mathcal{A}_K & \mathcal{B}_K \\ \mathcal{C}_K & \mathcal{D}_K \end{bmatrix} \begin{bmatrix} V_{\text{m}} \\ I_{\text{m}} \end{bmatrix} = \begin{bmatrix} \mathcal{A}_K & \mathcal{B}_K \\ \mathcal{C}_K & \mathcal{D}_K \end{bmatrix} \begin{bmatrix} \mathcal{A}_L & \mathcal{B}_L \\ \mathcal{C}_L & \mathcal{D}_L \end{bmatrix} \begin{bmatrix} V_{\text{out}} \\ I_{\text{out}} \end{bmatrix}$$

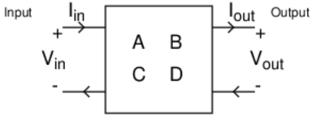
So the ABCD matrix of the combined network is

$$\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} = \begin{bmatrix} \mathcal{A}_K & \mathcal{B}_K \\ \mathcal{C}_K & \mathcal{D}_K \end{bmatrix} \begin{bmatrix} \mathcal{A}_L & \mathcal{B}_L \\ \mathcal{C}_L & \mathcal{D}_L \end{bmatrix}$$

which is nothing but the product of the ABCD matrices of the component networks from left to right.

This makes the ABCD matrix very useful in studying practical networks made by cascading simpler networks.

ABCD Matrix Elements



$V_{\text{in}} = \mathcal{A} V_{\text{out}} + \mathcal{B} I_{\text{out}}$
$I_{\text{in}} = \mathcal{C} V_{\text{out}} + \mathcal{D} I_{\text{out}}$

Measurement definitions:

$\mathcal{A} = \left.\frac{V_{\text{in}}}{V_{\text{out}}}\right|_{I_{\text{out}}=0}$, and $\mathcal{B} = \left.\frac{V_{\text{in}}}{I_{\text{out}}}\right|_{V_{\text{out}}=0}$.

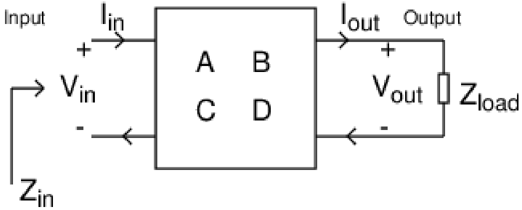
$\mathcal{C} = \left.\frac{I_{\text{in}}}{V_{\text{out}}}\right|_{I_{\text{out}}=0}$, and $\mathcal{D} = \left.\frac{I_{\text{in}}}{I_{\text{out}}}\right|_{V_{\text{out}}=0}$.

Dimensions: $\mathcal{A}$ and $\mathcal{D}$ are dimensionless. $\mathcal{B}$ is an impedance. $\mathcal{C}$ is an admittance.

Note that $\mathcal{A}$ and $\mathcal{C}$ are measured with the output open circuited, while $\mathcal{B}$ and $\mathcal{D}$ are measured with the output short circuited.

Note: The open circuit transfer function $T(s) = \left.\frac{V_{\text{out}}}{V_{\text{in}}}\right|_{I_{\text{out}}=0} = \frac{1}{\mathcal{A}}$.

Impedance Transformation



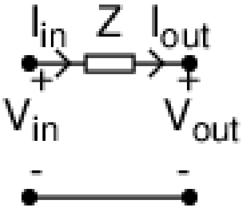
$$Z_{\text{in}} = \frac{V_{\text{in}}}{I_{\text{in}}} = \frac{\mathcal{A}V_{\text{out}} + \mathcal{B}I_{\text{out}}}{\mathcal{C}V_{\text{out}} + \mathcal{D}I_{\text{out}}} = \frac{\mathcal{A}V_{\text{out}}/I_{\text{out}} + \mathcal{B}}{\mathcal{C}V_{\text{out}}/I_{\text{out}} + \mathcal{D}} = \frac{\mathcal{A}Z_{\text{load}} + \mathcal{B}}{\mathcal{C}Z_{\text{load}} + \mathcal{D}}$$

since $V_{\text{out}}/I_{\text{out}} = Z_{\text{load}}$.

Möbius transformation or linear fractional transformation.

Where else do you see such transformations?

Series Element



Note: The element **must** be written as an **impedance.**

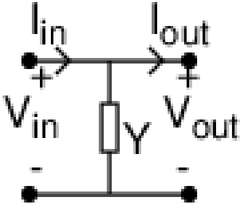
$V_{\text{in}} = V_{\text{out}} + ZI_{\text{out}}$

$I_{\text{in}} = I_{\text{out}} = 0V_{\text{out}} + I_{\text{out}}$

$\Rightarrow \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} = \begin{bmatrix} 1 & Z \\ 0 & 1 \end{bmatrix}$

What is the determinant of this matrix?

Shunt Element



Note: The element **must** be written as an **admittance.**

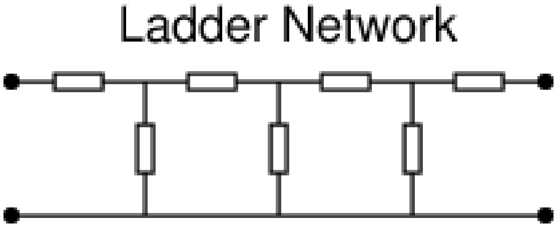
$V_{\text{in}} = V_{\text{out}} = V_{\text{out}} + 0I_{\text{out}}$

$I_{\text{in}} = YV_{\text{out}} + I_{\text{out}}$

$\Rightarrow \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ Y & 1 \end{bmatrix}$

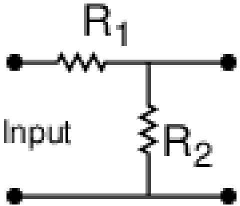
What is the determinant of this matrix?

Ladder Network as a Cascade



A ladder network can be considered as a cascade of series and shunt elements.

The Voltage Divider



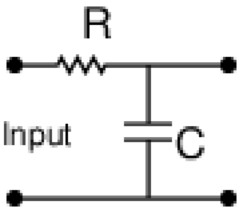
The ABCD matrix of this network is

$$\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} = \begin{bmatrix} 1 & R_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1/R_2 & 1 \end{bmatrix} = \begin{bmatrix} 1 + R_1/R_2 & R_1 \\ 1/R_2 & 1 \end{bmatrix}$$

Verify that $T(s) = \frac{1}{\mathcal{A}} = \frac{R_2}{R_1+R_2}$.

Note that for the shunt resistor, the entry in the matrix was for the $\mathcal{C}$ element, and was converted to the admittance $1/R_2$ first.

The RC Lowpass Filter



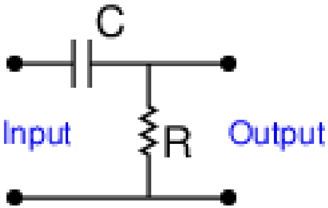
The ABCD matrix of this network is

$$\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} = \begin{bmatrix} 1 & R \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ sC & 1 \end{bmatrix} = \begin{bmatrix} 1 + sRC & R \\ sC & 1 \end{bmatrix}$$

Verify that $T(s) = \frac{1}{\mathcal{A}} = \frac{1}{1+sRC} = \frac{\frac{1}{RC}}{s+\frac{1}{RC}} = \frac{\omega_0}{s+\omega_0}$,

where $\omega_0 = \frac{1}{RC}$.

The CR Highpass Filter



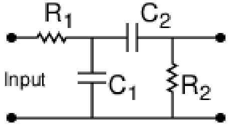
The ABCD matrix of this network is

$$\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{sC} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{1}{R} & 1 \end{bmatrix} = \begin{bmatrix} 1 + \frac{1}{sRC} & \frac{1}{sC} \\ \frac{1}{R} & 1 \end{bmatrix}$$

Verify that $T(s) = \frac{1}{\mathcal{A}} = \frac{1}{1+\frac{1}{sRC}} = \frac{s}{s+\frac{1}{RC}} = \frac{s}{s+\omega_0}$,

where $\omega_0 = \frac{1}{RC}$.

A Bandpass Filter



The ABCD matrix of this network is

$$\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} = \begin{bmatrix} 1 & R_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ sC_1 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{sC_2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{1}{R_2} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + sR_1C_1 & R_1 \\ sC_1 & 1 \end{bmatrix} \begin{bmatrix} 1 + \frac{1}{sR_2C_2} & \frac{1}{sC_2} \\ \frac{1}{R_2} & 1 \end{bmatrix}$$

We only write down the $\mathcal{A}$ element of the resulting matrix.

$\mathcal{A} = sR_1C_1 + 1 + \frac{R_1}{R_2} + \frac{R_1C_1}{R_2C_2} + \frac{1}{sR_2C_2}$.

At what frequency is $T(s) = \frac{1}{\mathcal{A}}$ real?

Answer: $f_0 = \frac{1}{2\pi\sqrt{R_1R_2C_1C_2}}$

What is $T(s)$ at that frequency?

Answer: $1/(1 + \frac{R_1}{R_2} + \frac{R_1C_1}{R_2C_2})$.

SPICE Code

File rccr.cir:

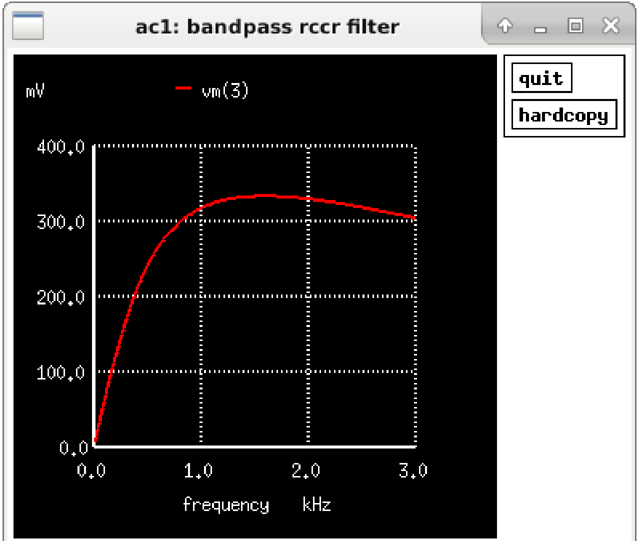
```
Bandpass RCCR Filter
*****************************
VIN 1 0 AC 1
R1 1 2 10k
C1 2 0 10n
C2 2 3 10n
R2 3 0 10k
.AC LIN 1000 10 3k

.control
run
plot vm(3)
plot vp(3)
.endcontrol
.END
```

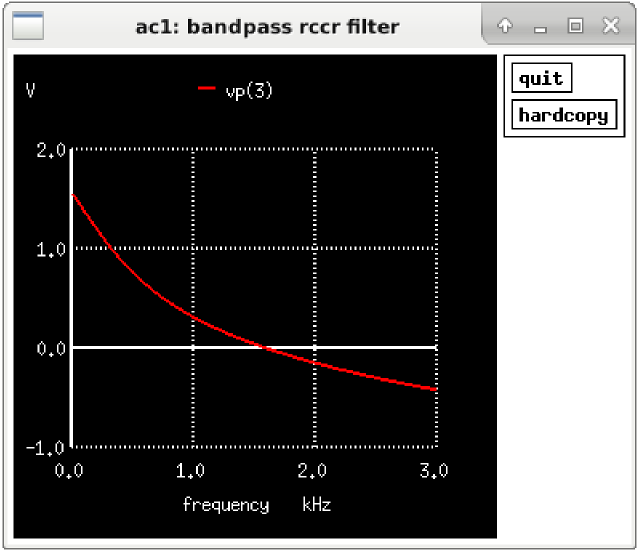
SPICE Results: Magnitude Plot

On Linux, you can type

ngspice rccr.cir



SPICE Results: Phase Plot



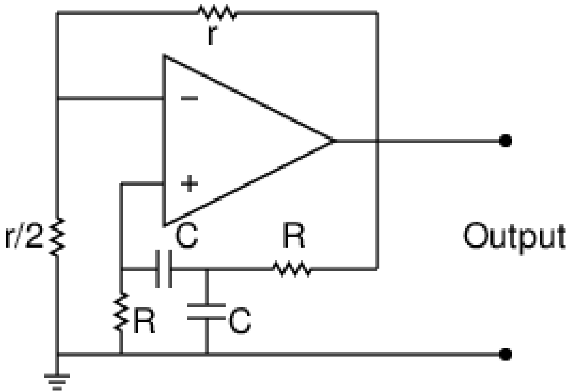
Available SPICE Software

- ngspice for Linux and OpenBSD (Recommended)
- LTspice for Windows

Making a Sinewave Oscillator

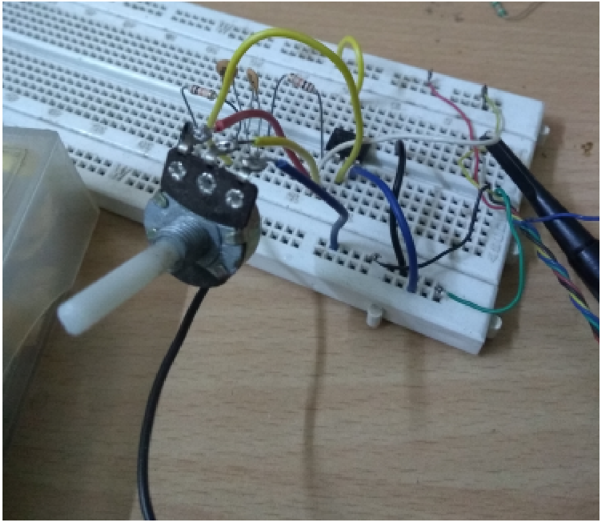
Let $R_1 = R_2 = R$, and $C_1 = C_2 = C$ in the circuit discussed.
Then $f_0 = \frac{1}{2\pi RC}$.
If $R = 10\,\mathrm{k}\Omega$, and $C = 10\,\mathrm{nF}$, $f_0 = 1.591\,55\,\mathrm{kHz}$.
$T(s)$ at this frequency is $1/3$.
So if we make a voltage amplifier of gain $+3$, we may be able to make a sinewave oscillator if we use this circuit in the feedback path.

Circuit Diagram: No AGC

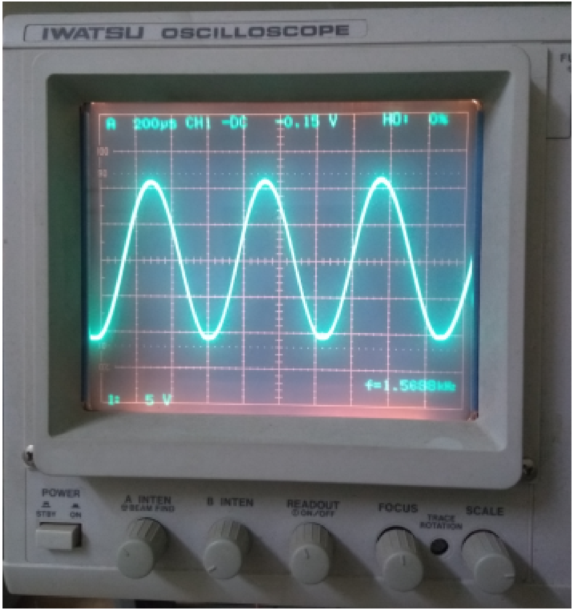


Will either fail to oscillate or give clipped output.

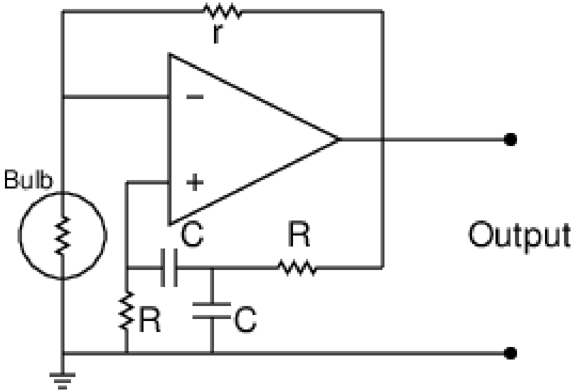
Bad Circuit



Bad Output: Clipped output

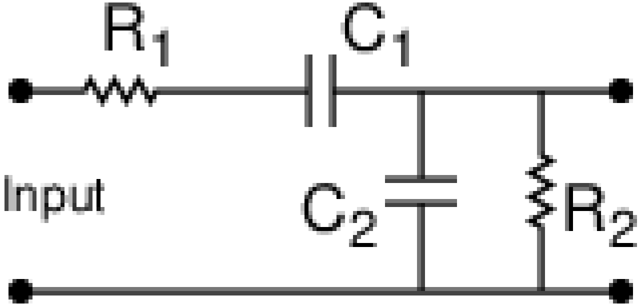


Circuit Diagram: With AGC



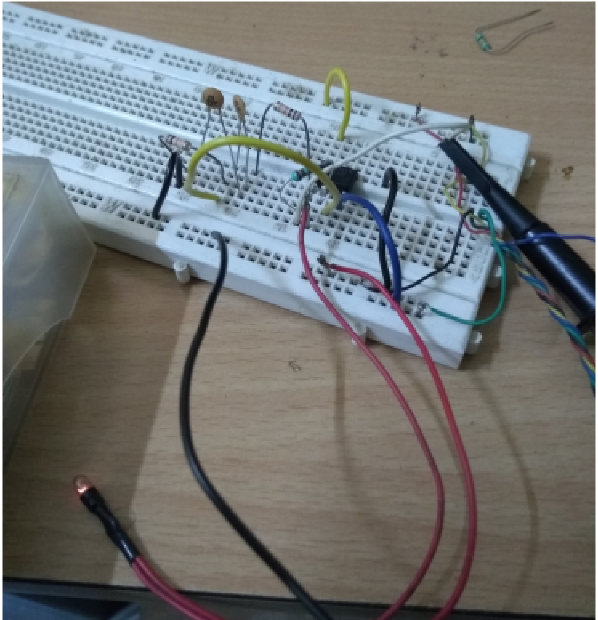
Can be made to work very well.
The success of Hewlett-Packard HP200A!
Note: HP200A uses a Wien bridge circuit which is slightly different.

Wien Bridge Circuit

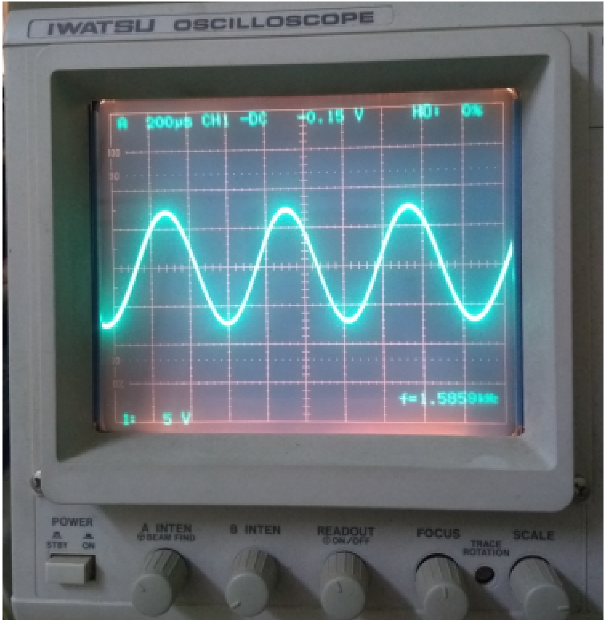


Note: Not used in our circuit.

Good Circuit



Good Output: No clipping



Summary

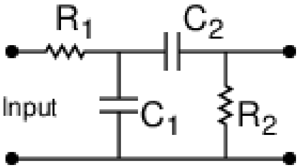
The ABCD matrix . . .

- . . . simplifies circuit analysis.
- . . . will often be used in this course.

Filters

- Used for selecting some frequencies and rejecting others.
- Some common uses:
  - Reducing noise
  - Frequency division multiplexing
  - Enhancing one harmonic of a periodic signal
- Our plan:
  1. Study simple filters or building blocks
  2. Combine these building blocks to make more complex filters

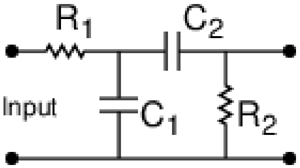
Example 1: RCCR BPF



Recall that
$\mathcal{A} = sR_1C_1 + 1 + \frac{R_1}{R_2} + \frac{R_1C_1}{R_2C_2} + \frac{1}{sR_2C_2}$.
So

$$T(s) = \frac{1}{\mathcal{A}} = \frac{\frac{1}{R_1C_1}s}{s^2 + \frac{1+\frac{R_1}{R_2}+\frac{R_1C_1}{R_2C_2}}{R_1C_1}s + \frac{1}{R_1R_2C_1C_2}}$$

Example 1: RCCR BPF $T(s)$ in Standard Form



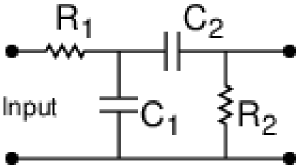
Comparing with the standard form we see that

$$\omega_0 = \frac{1}{\sqrt{R_1 R_2 C_1 C_2}}$$

$$Q = \frac{\sqrt{\frac{R_1 C_1}{R_2 C_2}}}{1 + \frac{R_1}{R_2} + \frac{R_1 C_1}{R_2 C_2}}$$

$$H = \frac{1}{1 + \frac{R_1}{R_2} + \frac{R_1 C_1}{R_2 C_2}}$$

Example 1: Special Case



If $R_1 = R_2 = R$, and $C_1 = C_2 = C$, we have

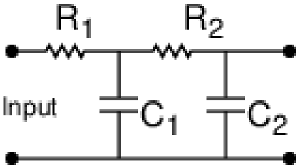
$$\omega_0 = \frac{1}{RC}$$

$$Q = \frac{1}{3}$$

$$H = \frac{1}{3}$$

Not very selective at all!

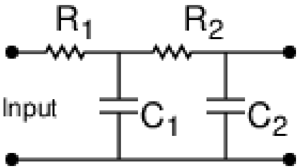
Example 2: RCRC LPF



Compute the ABCD matrix of the network to show that
$\mathcal{A} = s^2 R_1 R_2 C_1 C_2 + (R_1 C_1 + R_1 C_2 + R_2 C_2)s + 1$.
So

$$T(s) = \frac{1}{\mathcal{A}} = \frac{\frac{1}{R_1 R_2 C_1 C_2}}{s^2 + \frac{R_1 C_1 + R_1 C_2 + R_2 C_2}{R_1 R_2 C_1 C_2} s + \frac{1}{R_1 R_2 C_1 C_2}}$$

Example 2: RCRC LPF $T(s)$ in Standard Form



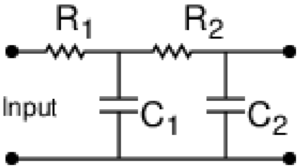
Comparing with the standard form we see that

$$\omega_0 = \frac{1}{\sqrt{R_1 R_2 C_1 C_2}}$$

$$Q = \frac{\sqrt{R_1 R_2 C_1 C_2}}{R_1 C_1 + R_1 C_2 + R_2 C_2}$$

$$H = 1$$

Example 2: Special Case



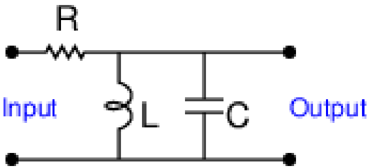
If $R_1 = R_2 = R$, and $C_1 = C_2 = C$, we have

$$\omega_0 = \frac{1}{RC}$$

$$Q = \frac{1}{3}$$

$$H = 1$$

Example 3: BPF from the Parallel RLC Network



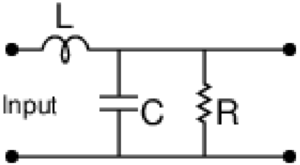
$$T(s) = \frac{\frac{s}{RC}}{s^2 + \frac{s}{RC} + \frac{1}{LC}}$$

Comparing with the standard form we see that $\omega_0 = \frac{1}{\sqrt{LC}}$, $Q = \frac{R}{\sqrt{L/C}}$, and $H = 1$.

Note that the expression for Q here differs from the expression that was derived for the BPF based on the series RLC circuit.

Here Q is proportional to R, there it was inversely proportional to R.

Example 4: LPF from the Parallel RLC Network

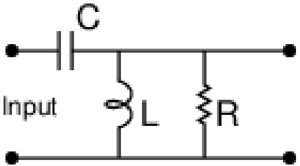


$$T(s) = \frac{\frac{1}{LC}}{s^2 + \frac{s}{RC} + \frac{1}{LC}}$$

Comparing with the standard form we see that $\omega_0 = \frac{1}{\sqrt{LC}}$, $Q = \frac{R}{\sqrt{L/C}}$, and $H = 1$.

This circuit is used for impedance matching in industrial applications.

Example 5: HPF from the Parallel RLC Network



$$T(s) = \frac{s^2}{s^2 + \frac{s}{RC} + \frac{1}{LC}}$$

Comparing with the standard form we see that $\omega_0 = \frac{1}{\sqrt{LC}}$, $Q = \frac{R}{\sqrt{L/C}}$, and $H = 1$.

This circuit is also used for impedance matching in industrial applications.

Why RC Filters?

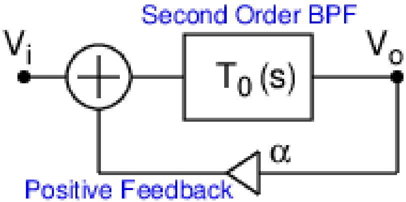
- Inductors are practical at high frequencies.
- At audio frequencies, inductors tend to be bulkier and more expensive compared to resistors and capacitors.
- So there is a desire to make audio frequency filters using resistors and capacitors only.
- But, passive RC second order networks seem to have low Q.
- How can we get more Q?
- The answer is the *active filter*.
- Active filters use amplification to compensate for the losses.

Q-Enhancement using Positive Feedback

Consider a second order BPF whose transfer function is

$$T_0(s) = \frac{H_0 \frac{\omega_{00}}{Q_0} s}{s^2 + \frac{\omega_{00}}{Q_0} s + \omega_{00}^2}$$

The extra 0s in the subscripts are there to indicate original parameters.
Now let us use positive feedback as shown.



What is the new transfer function?

New Transfer Function

$V_o = T_0(s)(V_i + \alpha V_o)$
$V_o(1 - \alpha T_0(s)) = T_0(s) V_i$

$$T(s) = \frac{V_o}{V_i} = \frac{T_0(s)}{1 - \alpha T_0(s)}$$

Substitution of the expression for $T_0(s)$ and simplification gives us

$$T(s) = \frac{H_0 \frac{\omega_{00}}{Q_0} s}{s^2 + (1 - \alpha H_0)\frac{\omega_{00}}{Q_0} s + \omega_{00}^2}$$

New Parameters

Comparing with the standard form

$$T(s) = \frac{H\frac{\omega_0}{Q}s}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$$

we see the following.
$\omega_0 = \omega_{00}$. Centre angular frequency does NOT change.

$$Q = \frac{Q_0}{1 - \alpha H_0}$$

$$H = \frac{H_0}{1 - \alpha H_0}$$

Both Q and H are *enhanced* by the factor $\frac{1}{1-\alpha H_0}$.
We need to be careful. If αH_0 exceeds 1, the circuit will oscillate.

Q-Enhancement

The positive feedback scheme that was described can be implemented using two operational amplifiers.

In practice, only one operational amplifier may be enough.

Not only second order BPF, even second order LPF and HPF circuits can have their Q enhanced using amplifiers.