PH 207
Simple RC, RL, and RLC Filters

A. Mohanty

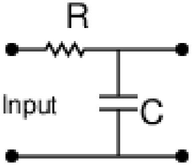
Department of Instrumentation and Applied Physics
Indian Institute of Science
Bangalore 560012

January 28, 2025

Simple RLC Circuits as Building Blocks

1. First order RC or RL circuits.
2. Second order RLC circuits.
3. Second order RC circuits. (To be discussed in the next class.)

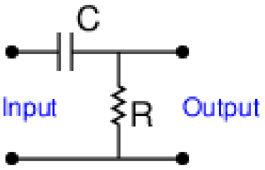
RC LPF



$$T(s) = \frac{\omega_0}{s + \omega_0}$$

where, $\omega_0 = \frac{1}{RC}$

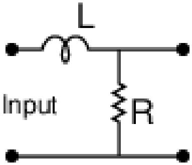
CR HPF



$$T(s) = \frac{s}{s + \omega_0}$$

where, $\omega_0 = \frac{1}{RC}$

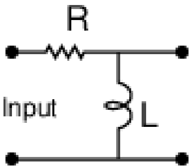
LR LPF



$$T(s) = \frac{\omega_0}{s + \omega_0}$$

where, $\omega_0 = \frac{R}{L}$

RL HPF



$$T(s) = \frac{s}{s + \omega_0}$$

where, $\omega_0 = \frac{R}{L}$

First Order LPF Transfer Function

$$T(s) = \frac{\omega_0}{s + \omega_0}$$

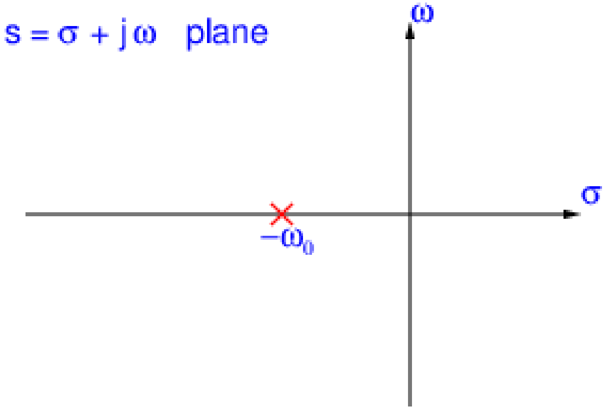
$$T(j\omega) = \frac{1}{1 + j\omega/\omega_0}$$

$$|T(j\omega)| = \frac{1}{\sqrt{1 + (\omega/\omega_0)^2}}$$

$$\underline{/T(j\omega)} = -\arctan(\omega/\omega_0).$$

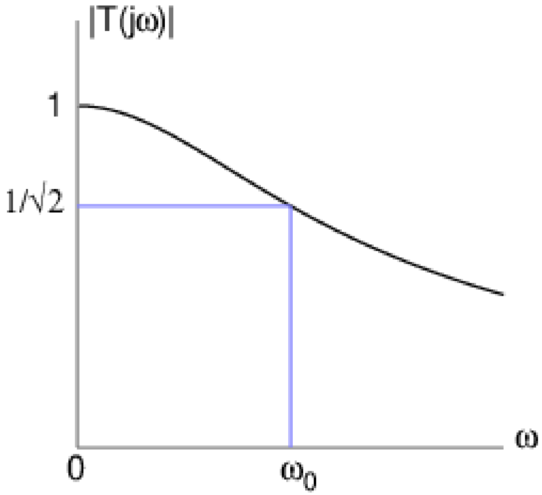
So $|T(j\omega_0)| = 1/\sqrt{2}$.
For $|\omega| \gg \omega_0$, $|T(j\omega)| \approx \omega_0/|\omega|$.

First Order LPF Pole-zero Diagram

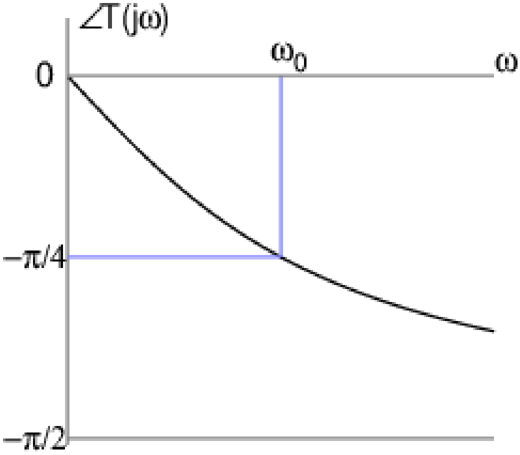


Has one pole and no zero.

First Order LPF TF Magnitude Plot



First Order LPF TF Phase Plot



First Order HPF Transfer Function

$$T(s) = \frac{s}{s+\omega_0}.$$

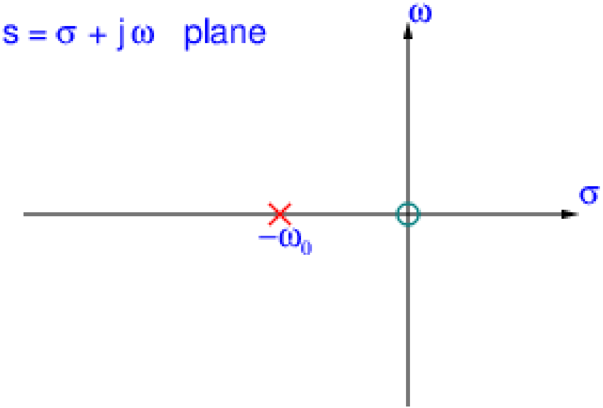
$$T(j\omega) = \frac{1}{1 - j\omega_0/\omega}.$$

$$|T(j\omega)| = \frac{1}{\sqrt{1+(\omega_0/\omega)^2}}.$$

$$\underline{/T(j\omega)} = \arctan(\omega_0/\omega) = \pi/2 - \arctan(\omega/\omega_0).$$

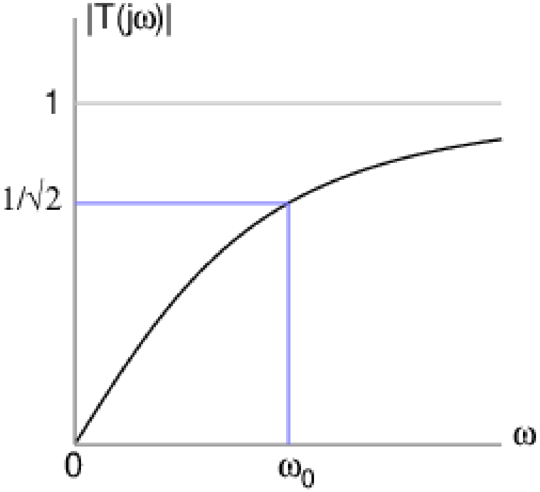
So $|T(j\omega_0)| = 1/\sqrt{2}$.
For $|\omega| \ll \omega_0$, $|T(j\omega)| \approx |\omega|/\omega_0$.

First Order HPF Pole-zero Diagram

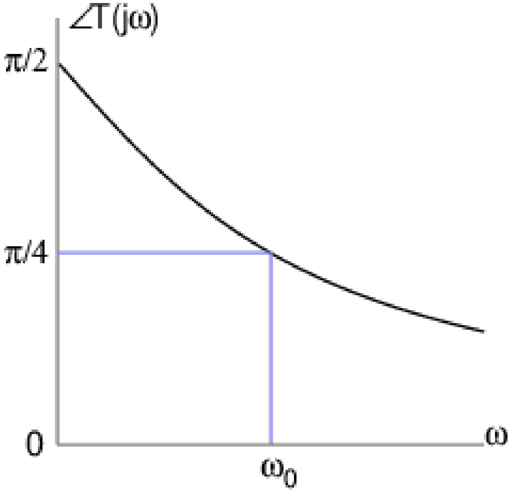


Has one pole and one zero.

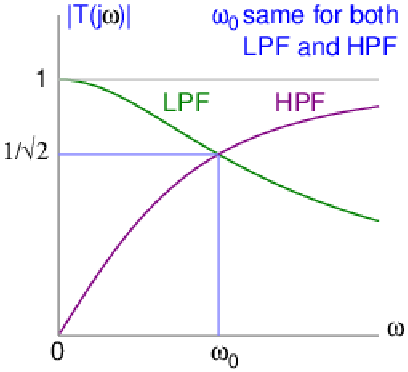
First Order HPF TF Magnitude Plot



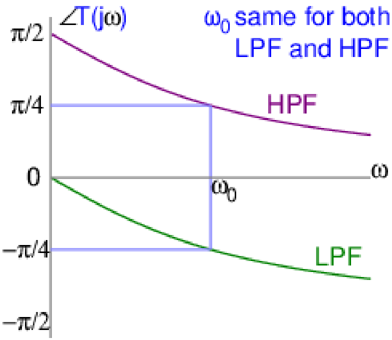
First Order HPF TF Phase Plot



First Order LPF and HPF TF Magnitude Plots



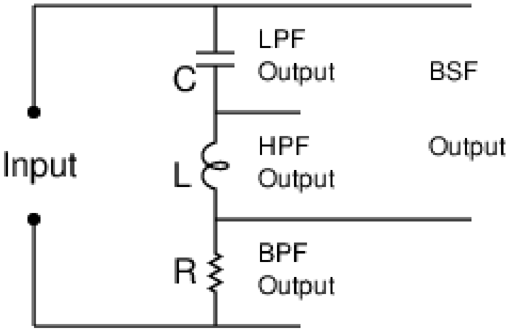
First Order LPF and HPF TF Phase Plots



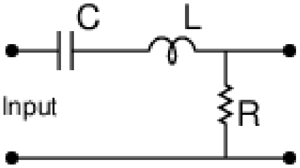
Comparison of first order LPF and HPF having the same ω_0:

- Phase shift is often easier to measure.
- Both LPF and HPF have similar looking phase plots.
- HPF phase leads LPF phase by a right angle.

The Series RLC Circuit



The Series RLC Bandpass Filter



$$T(s) = \frac{R}{sL + R + \frac{1}{sC}}$$

Simplify to get

$$T(s) = \frac{\frac{R}{L}s}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

The Second Order Bandpass Transfer Function

$$T(s) = \frac{\frac{R}{L}s}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Write $\frac{1}{LC} = \omega_0^2$, and $\frac{R}{L} = 2\alpha$ to get

$$T(s) = \frac{2\alpha s}{s^2 + 2\alpha s + \omega_0^2}$$

For small loss, that is for small R, or for small α, $T(s)$ has poles at $-\alpha \pm j\sqrt{\omega_0^2 - \alpha^2}$.
So α is the decay constant.
ω_0 is the angular frequency of oscillations for no loss.

Magnitude Response in the Frequency Domain

$$T(s) = \frac{2\alpha s}{s^2 + 2\alpha s + \omega_0^2}$$

$$T(j\omega) = \frac{j2\alpha\omega}{-\omega^2 + j2\alpha\omega + \omega_0^2} = \frac{1}{1 + \frac{\omega_0^2 - \omega^2}{j2\alpha\omega}} = \frac{1}{1 + j\frac{\omega^2 - \omega_0^2}{2\alpha\omega}}$$

Centre Angular Frequency

$$T(j\omega) = \frac{1}{1 + j\frac{\omega^2 - \omega_0^2}{2\alpha\omega}}$$

When is $|T(j\omega)| = 1$?
This happens when $\omega = \pm\omega_0$.
At other values of ω, $|T(j\omega)| < 1$.

Half-power Angular Frequencies

$$T(j\omega) = \frac{1}{1 + j\frac{\omega^2 - \omega_0^2}{2\alpha\omega}}$$

When is $|T(j\omega)| = \frac{1}{\sqrt{2}}$?

This happens when $\frac{\omega^2 - \omega_0^2}{2\alpha\omega} = \pm 1$.

Or, $\omega^2 - \omega_0^2 = \pm 2\alpha\omega$.

The two quadratic equations are,

$\omega^2 - 2\alpha\omega - \omega_0^2 = 0$,

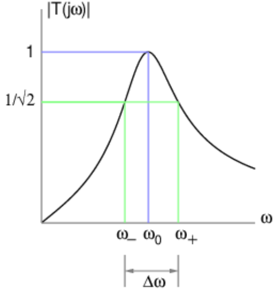
and

$\omega^2 + 2\alpha\omega - \omega_0^2 = 0$.

The positive root of the first quadratic equation is $\omega_+ = \alpha + \sqrt{\alpha^2 + \omega_0^2}$.

The positive root of the second quadratic equation is $\omega_- = -\alpha + \sqrt{\alpha^2 + \omega_0^2}$.

Magnitude Plot of the BPF Transfer Function



Note that $\omega_+\omega_- = \omega_0^2$. Half-power angular bandwidth: $\Delta\omega = \omega_+ - \omega_- = 2\alpha$.
Quality factor

$$Q = \frac{\omega_0}{\Delta\omega} = \frac{\omega_0}{2\alpha}$$

What is Q?

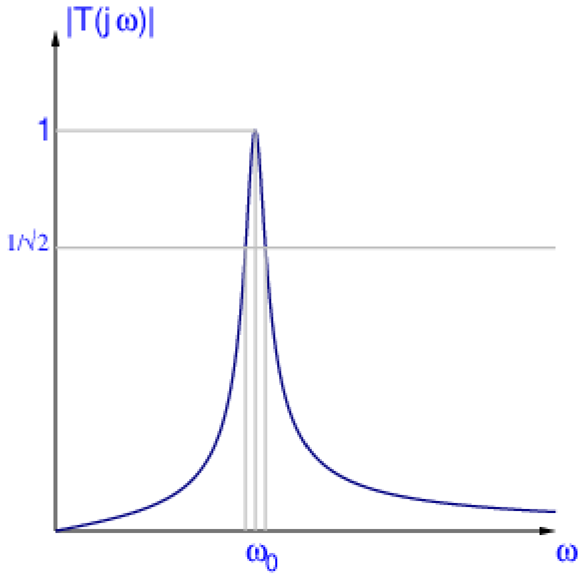
Q is a measure of the selectivity of the BPF. Note that this definition in the frequency domain is the original, exact definition of Q.
Note that $2\alpha = \Delta\omega = \frac{\omega_0}{Q}$.

$$\omega_+ = \left(\sqrt{1 + \frac{1}{4Q^2}} + \frac{1}{2Q}\right)\omega_0$$

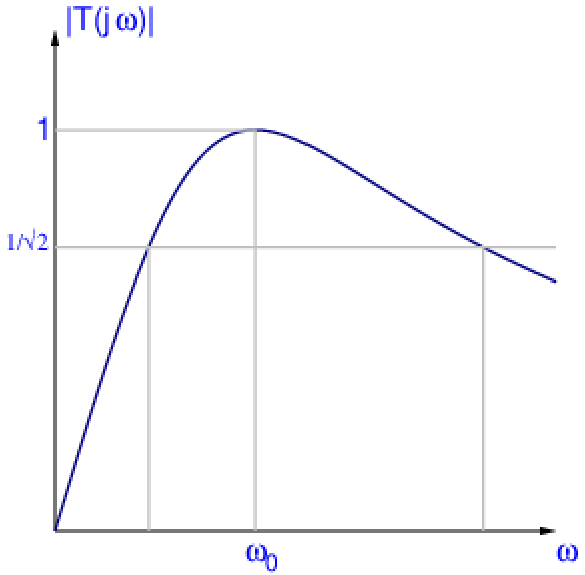
$$\omega_- = \left(\sqrt{1 + \frac{1}{4Q^2}} - \frac{1}{2Q}\right)\omega_0$$

Remember that ω_0 is the *geometric* mean of ω_+ and ω_-.
It is NOT the arithmetic mean of ω_+ and ω_-.

$|T(j\omega)|$ for $Q = 10$



$|T(j\omega)|$ for $Q = 0.6$



BPF Phase

$$T(j\omega) = \frac{2\alpha j\omega}{2\alpha j\omega + \omega_0^2 - \omega^2} = \frac{j\omega\omega_0/Q}{j\omega\omega_0/Q + \omega_0^2 - \omega^2} = \frac{1}{1 + jQ\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)}. \tag{1}$$

Phase angle is

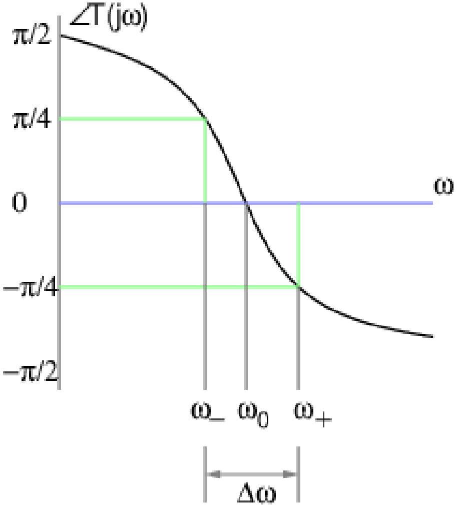
$$\underline{/T(j\omega)} = \arctan\left(Q\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)\right). \tag{2}$$

Special values:

- $\underline{/T(j0)} = \pi/2$.
- $\underline{/T(j\omega_0)} = 0$.
- $\underline{/T(j\infty)} = -\pi/2$.
- $\underline{/T(j\omega_-)} = \pi/4$.
- $\underline{/T(j\omega_+)} = -\pi/4$.

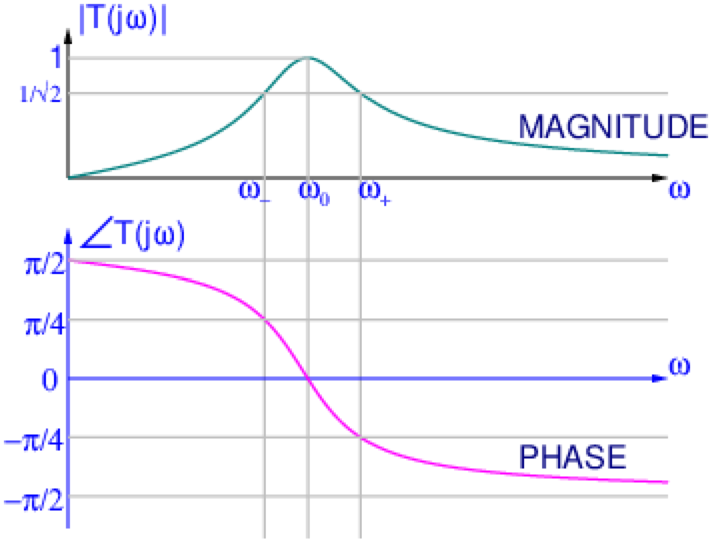
Phase is important because it is often easier to measure.

Phase Plot of the BPF Transfer Function



Phase is easier to measure!

BPF magnitude and phase on the same plot



BPF MAGNITUDE AND PHASE PLOTS FOR Q = 2.5

BPF Transfer Function Rewritten

$$T(s) = \frac{2\alpha s}{s^2 + 2\alpha s + \omega_0^2}$$

Since $2\alpha = \frac{\omega_0}{Q}$,

$$T(s) = \frac{\frac{\omega_0}{Q} s}{s^2 + \frac{\omega_0}{Q} s + \omega_0^2}$$

This is the standard form of the transfer function of the BPF.
For the series RLC BPF,

$$\omega_0 = 1/\sqrt{LC},$$

and

$$Q = \frac{\omega_0}{2\alpha} = \frac{\frac{1}{\sqrt{LC}}}{\frac{R}{L}} = \frac{\sqrt{L/C}}{R}.$$

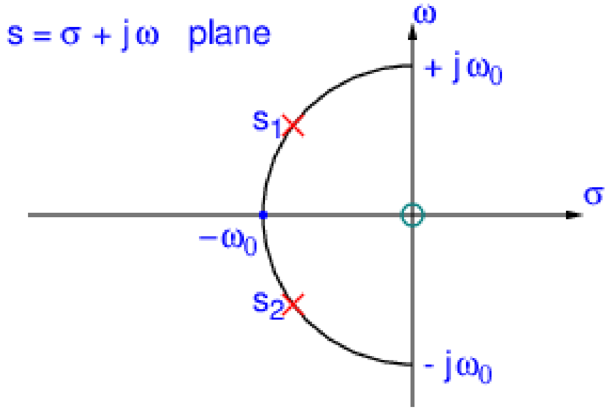
For other circuits or physical systems, these expressions will need to be determined in terms of the parameters of that system.

General Second Order BPF Transfer Function

$$T(s) = \frac{H\frac{\omega_0}{Q}s}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$$

ω_0: Centre angular frequency
Q: Quality factor
H: Gain factor

Second Order BPF Pole-zero Diagram



Shown for $Q > \frac{1}{2}$. Has two poles and one zero.

Second Order BPF Pole Locations

Find zeros of $s^2 + \frac{\omega_0}{Q} s + \omega_0^2$.

Case $Q > \frac{1}{2}$ (Underdamped)

$$s_1 = -\frac{\omega_0}{2Q} + j\omega_0 \sqrt{1 - \frac{1}{4Q^2}}$$

$$s_2 = -\frac{\omega_0}{2Q} - j\omega_0 \sqrt{1 - \frac{1}{4Q^2}}$$

Complex conjugate pair of poles. $s_1 s_2 = \omega_0^2$.

Case $Q = \frac{1}{2}$ (Critically damped)

$$s_1 = s_2 = -\omega_0.$$

Equal, negative real poles.

Second Order BPF Pole Locations

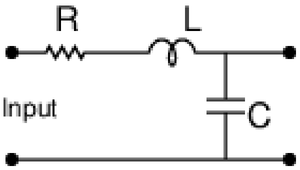
Case $Q < \frac{1}{2}$ (Overdamped)

$$s_1 = -\frac{\omega_0}{2Q} + \omega_0\sqrt{\frac{1}{4Q^2} - 1}$$

$$s_2 = -\frac{\omega_0}{2Q} - \omega_0\sqrt{\frac{1}{4Q^2} - 1}$$

Unequal negative real poles. $s_1 s_2 = \omega_0^2$.

The Series RLC Lowpass Filter



$$T(s) = \frac{\frac{1}{sC}}{sL + R + \frac{1}{sC}}$$

Simplify to get

$$T(s) = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} = \frac{\omega_0^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$$

Second Order LPF Magnitude Response

$$T(j\omega) = \frac{\omega_0^2}{\omega_0^2 - \omega^2 + j\frac{\omega\omega_0}{Q}}$$

At what frequency is $|T(j\omega)|$ maximum?
The numerator is constant. The square of the magnitude of the denominator is

$$(\omega_0^2 - \omega^2)^2 + \frac{\omega^2\omega_0^2}{Q^2} = \omega_0^4 + \omega^4 - 2\omega_0^2\omega^2 + \frac{\omega^2\omega_0^2}{Q^2}$$

$$= \omega_0^4 + \omega^4 - 2\omega_0^2\omega^2\left(1 - \frac{1}{2Q^2}\right)$$

We will try to complete squares here. The result depends on the value of Q.

Magnitude Response (continued)

If $Q \leq 1/\sqrt{2}$, all terms are non-negative and the denominator is an increasing function of ω.
In that case, $|T(j\omega)|$ has a maximum value of 1 at $\omega = 0$. For any other ω, $|T(j\omega)|$ is a monotonically decreasing function of $|\omega|$. We then say that there is *no peaking*.

Magnitude Response (continued)

If $Q > 1/\sqrt{2}$, we can complete the square to get the denominator magnitude squared as

$$\left(\omega^2 - \omega_0^2\left(1 - \frac{1}{2Q^2}\right)\right)^2 + \omega_0^4 \frac{1}{Q^2}\left(1 - \frac{1}{4Q^2}\right)$$

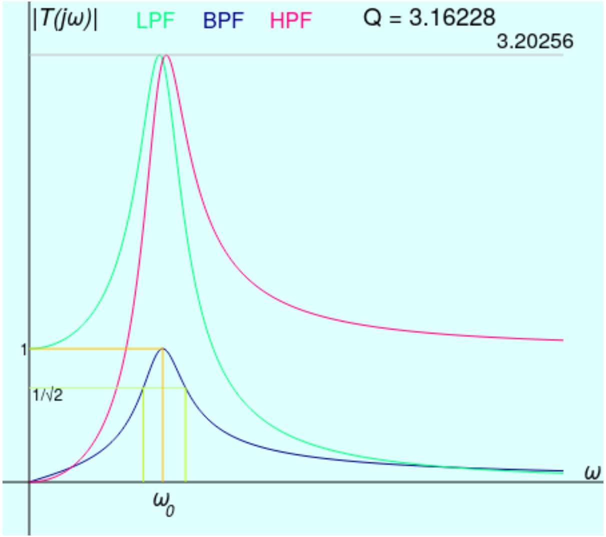
So $|T(j\omega)|$ is maximum when

$$|\omega| = \omega_L = \omega_0 \sqrt{1 - \frac{1}{2Q^2}}.$$

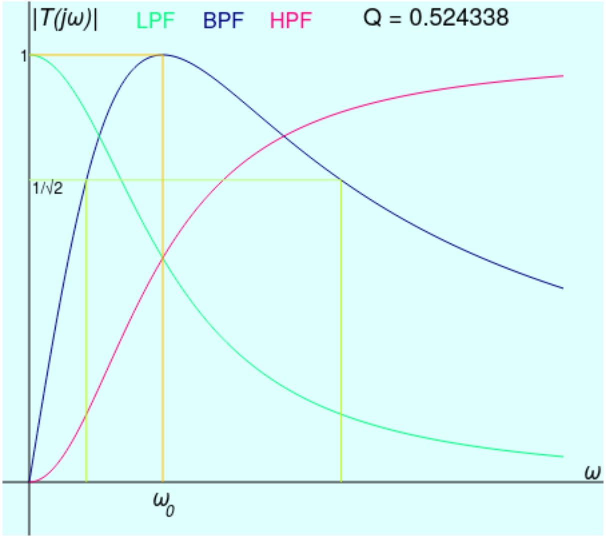
$$|T(j\omega_L)| = \frac{Q}{\sqrt{1 - \frac{1}{4Q^2}}}$$

This gives rise to *peaking*.

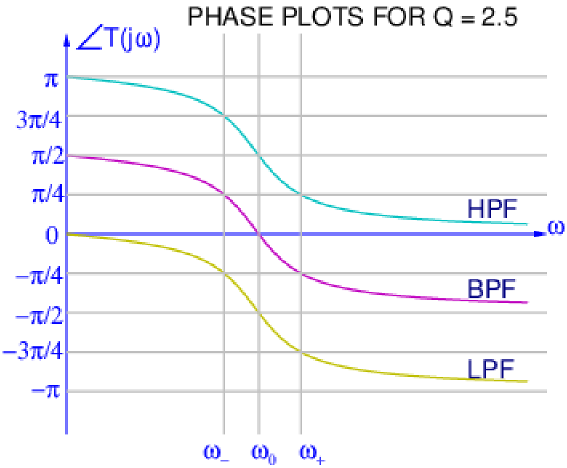
Case of *Peaking*



Case of *No Peaking*



$T(j\omega)$ Phase



Observations

Note that $T_{\mathrm{HPF}}(j\omega)/T_{\mathrm{BPF}}(j\omega) = jQ\omega/\omega_0$, and $T_{\mathrm{LPF}}(j\omega)/T_{\mathrm{BPF}}(j\omega) = -jQ\omega_0/\omega$.
So for positive ω, the HPF phase leads the BPF phase by $\pi/2$, while the LPF phase lags the BPF phase by $\pi/2$, as the plot shows.
In the same way, for the first-order case, HPF phase leads the LPF phase by $\pi/2$.
Points to note:

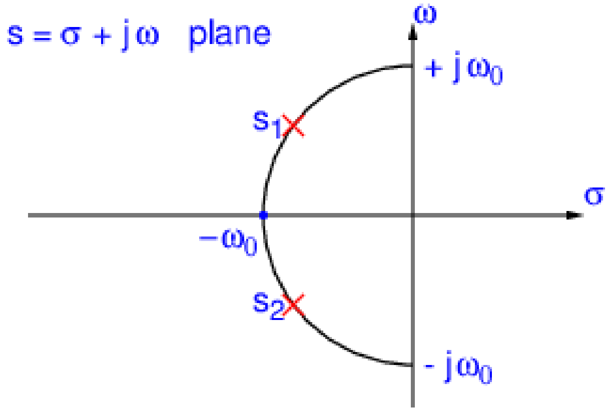
- Unlike the magnitude plots, the phase plots are monotonic.
- HPF, BPF, and LPF phase plots are very simply related to one another.
- Phase is often easier to measure.

General Second Order LPF Transfer Function

$$T(s) = \frac{H\omega_0^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$$

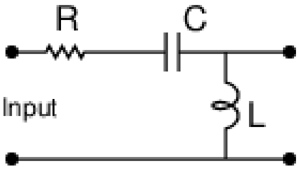
ω_0: Centre angular frequency
Q: Quality factor
H: Gain factor

Second Order LPF Pole-zero Diagram



Shown for $Q > \frac{1}{2}$. Has two poles and no zero.

The Series RLC Highpass Filter



$$T(s) = \frac{sL}{sL + R + \frac{1}{sC}}$$

Simplify to get

$$T(s) = \frac{s^2}{s^2 + \frac{R}{L}s + \frac{1}{LC}} = \frac{s^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$$

HPF Magnitude Response

If $Q > 1/\sqrt{2}$, show that $|T(j\omega)|$ is maximum when

$$|\omega| = \omega_H = \frac{\omega_0}{\sqrt{1 - \frac{1}{2Q^2}}}.$$

$$|T(j\omega_H)| = \frac{Q}{\sqrt{1 - \frac{1}{4Q^2}}}$$

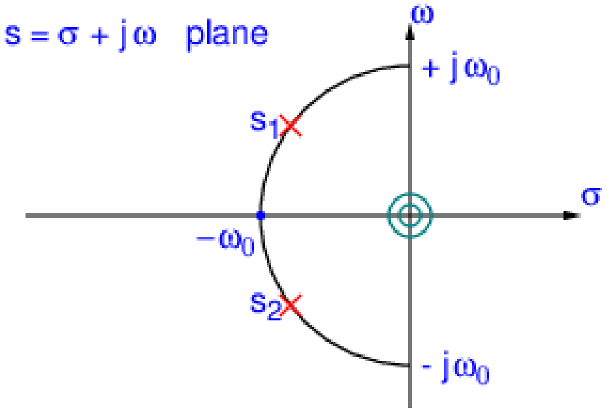
Note that $\omega_L \omega_H = \omega_0^2$, even though ω_H and ω_L refer to different types of filters.
If $Q \leq 1/\sqrt{2}$, then there is no peaking.

General Second Order HPF Transfer Function

$$T(s) = \frac{Hs^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$$

ω_0: Centre angular frequency
Q: Quality factor
H: Gain factor

Second Order HPF Pole-zero Diagram



Shown for $Q > \frac{1}{2}$. Has two poles and two zeros.

Notation and Terminology

Note that even though the second order LPF and HPF are not really bandpass filters, we still use the notations ω_0 and Q.
The meanings are different, even though the expressions are the same.

Broader Use of Q

Not all tuned systems are second order systems.
Still, the symbol Q is used in such systems.
One should be careful in such cases.

RLC Circuit Second-order Transfer Functions: LPF, BPF, and HPF

Finally, here are the second-order transfer functions connected with the series RLC circuit.
LPF (Lowpass Filter):

$$T_{\mathrm{LPF}}(s) = \frac{\omega_0^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}. \tag{3}$$

BPF (Bandpass Filter):

$$T_{\mathrm{BPF}}(s) = \frac{\frac{\omega_0}{Q}s}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}. \tag{4}$$

HPF (Highpass Filter):

$$T_{\mathrm{HPF}}(s) = \frac{s^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}. \tag{5}$$

For the series RLC circuit, $\omega_0 = 1/\sqrt{LC}$, and $Q = \frac{\sqrt{L/C}}{R}$.
When discussing a particular type of filter, the subscript of T may be omitted.

General Second-order Transfer Functions: LPF, BPF, and HPF

Due to the presence of attenuation or amplification, in general, a second-order filter will have a gain factor as shown in the following expressions.
LPF (Lowpass Filter):

$$T_{\mathrm{LPF}}(s) = \frac{H\omega_0^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}. \tag{6}$$

BPF (Bandpass Filter):

$$T_{\mathrm{BPF}}(s) = \frac{H\frac{\omega_0}{Q}s}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}. \tag{7}$$

HPF (Highpass Filter):

$$T_{\mathrm{HPF}}(s) = \frac{Hs^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}. \tag{8}$$

H is a gain factor that may be positive or negative.
Its magnitude may be smaller or greater than 1.