IN 221 (AUG) 3:0
Sensors and Transducers
Electromagnetic Sensors and Transducers
Lecture 6

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Energy Stored in a Capacitor

Charge stored:

$$Q = Cv. \tag{1}$$

Current:

$$i = \frac{\mathrm{d}Q}{\mathrm{d}t} = C\frac{\mathrm{d}v}{\mathrm{d}t}. \tag{2}$$

Power:

$$p = vi = Cv\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}\left(\frac{1}{2}Cv^2\right)}{\mathrm{d}t}. \tag{3}$$

Energy stored:

$$U = \frac{1}{2}Cv^2 = \frac{1}{2}C\left(\frac{Q}{C}\right)^2 = \frac{1}{2}\frac{Q^2}{C}. \tag{4}$$

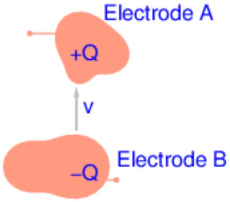
U was expressed in terms of charge Q, because in an isolated capacitor, Q is a constant.

Reconfiguration

What happens when we *reconfigure*, that is change the position and/or orientation, of the electrodes of a capacitor?

- The capacitance C, and the stored energy U change.
- If the capacitor is isolated, the stored charge Q does *not* change.
- This can be used to derive the *force* and the *torque* exterted by an electrode of a charged capacitor.
- In this study, we only derive an expression for the force.

Force exterted by an electrode

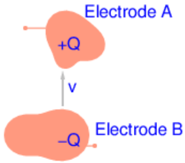


Capacitor Electrodes

The figure shows electrodes of a capacitor which is charged to charge Q. Irregular shapes are shown, because this is part of a general derivation that is not specific to any standard type of capacitor.

Let one of the electrodes, say Electrode A, be considered movable.

Force exterted by an electrode



Capacitor Electrodes

Quantities like C, and U are now functions of the position and the orientation of Electrode A.
Assume that the orientation is fixed. Let the position of Electrode A be specified by coordinates x, y, and z of marked point on it. Then

$$C = C(x, y, z), \tag{5}$$

and

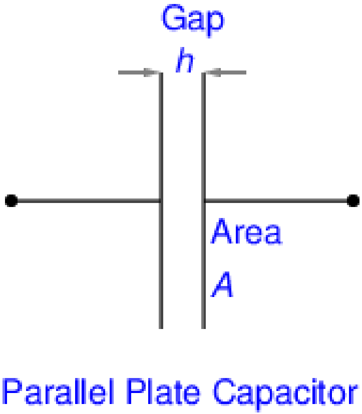
$$U = U(x, y, z). \tag{6}$$

Force exterted by an electrode

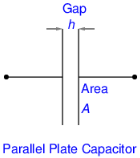
Let the force exterted by Electrode A, when it is held in place, be $\vec{F}$.
Changing the position of the electrode by a small displacement $\Delta\vec{r}$ would require work $-\vec{F}\cdot\Delta\vec{r}$ to be done on the system.
If the capacitor is isolated, this work would be added to the stored energy of the capacitor. So we have

$$\vec{F} = -\,\mathbf{grad}\,U = -\,\mathbf{grad}\left(\frac{1}{2}\frac{Q^2}{C}\right) = \frac{1}{2}\frac{Q^2}{C^2}\,\mathbf{grad}\,C = \frac{1}{2}v^2\,\mathbf{grad}\,C. \tag{7}$$

Force in a parallel plate capacitor



Force in a parallel plate capacitor



For the parallel plate capacitor shown,

$$C = \frac{\epsilon_0 A}{h}. \tag{8}$$

$$\textbf{grad}\, C = -\frac{\epsilon_0 A}{h^2}\hat{\mathbf{h}}, \tag{9}$$

rate of change of C in directions perpendicular to h being 0. On the electrode on the right,

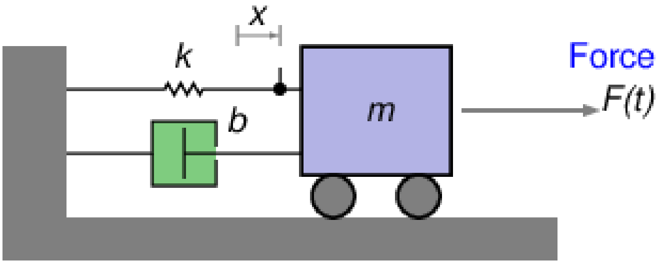
$$\vec{F} = \frac{1}{2}v^2\,\textbf{grad}\, C. = -\frac{1}{2}\frac{\epsilon_0 A v^2}{h^2}\hat{\mathbf{h}}. \tag{10}$$

the negative sign indicating a force to the left.

Spring-Mass-Dashpot System: Special case

We have neglected to mention the first-order systems which should be studied before the second-order systems.

Spring-Mass-Dashpot System: Modelling



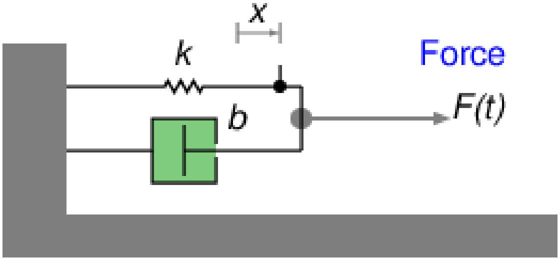
x: Displacement of the mass from its equilibrium position

$$m\ddot{x} + b\dot{x} + kx = F \tag{11}$$

F: Force
What happens when $m \to 0$?

$$b\dot{x} + kx = F \tag{12}$$

Spring-Dashpot System



$$b\dot{x} + kx = F \quad (13)$$

Study of two possible systems:

- Input is F, output is x, or to have the same dimension, $F_{\text{spring}} = kx$, tension force in the spring.
- Input is F, output is $\dot{x}$, or to have the same dimension, $F_{\text{dashpot}} = b\dot{x}$, tension force in the dashpot.

First-order LPF: Input F, output $F_{\text{spring}} = kx$

$$T(s) = \frac{k}{bs + k} = \frac{k/b}{s + k/b} = \frac{\omega_0}{s + \omega_0}, \tag{14}$$

where,

$$\omega_0 = k/b. \tag{15}$$

This is an example of a first-order lowpass filter.

First-order HPF: Input F, output $F_{\text{dashpot}} = b\dot{x}$

$$T(s) = \frac{bs}{bs + k} = \frac{s}{s + k/b} = \frac{s}{s + \omega_0}, \tag{16}$$

where,

$$\omega_0 = k/b. \tag{17}$$

This is an example of a first-order highpass filter.

First-order LPF

$$T(s) = \frac{\omega_0}{s + \omega_0}$$

where, $\omega_0 = k/b$.

First-order HPF

$$T(s) = \frac{s}{s + \omega_0}$$

where, $\omega_0 = k/b$.

First Order LPF Transfer Function

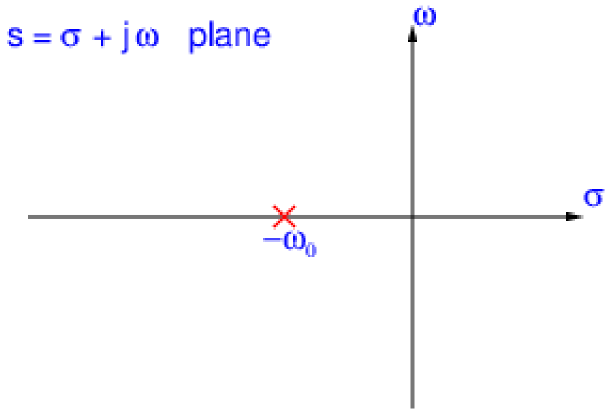
$$T(s) = \frac{\omega_0}{s + \omega_0}$$

$$T(j\omega) = \frac{1}{1 + j\omega/\omega_0}$$

$$|T(j\omega)| = \frac{1}{\sqrt{1 + (\omega/\omega_0)^2}}$$

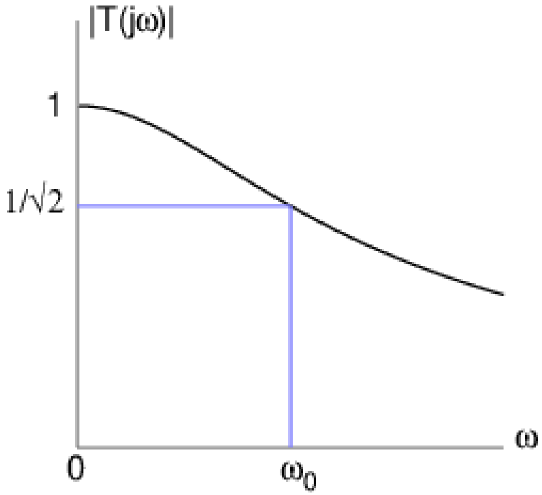
So $|T(j\omega_0)| = 1/\sqrt{2}$.
For $|\omega| \gg \omega_0$, $|T(j\omega)| \approx \omega_0/|\omega|$.

First Order LPF Pole-zero Diagram

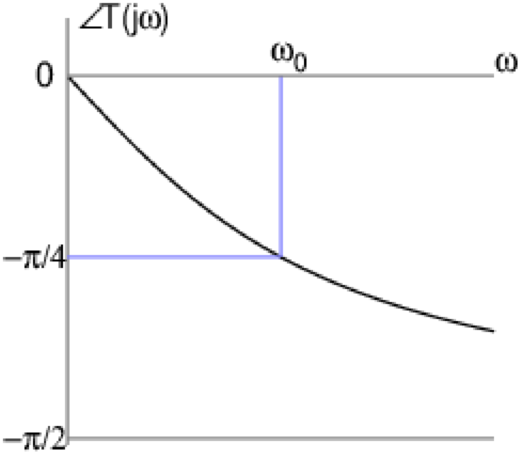


Has one pole and no zero.

First Order LPF TF Magnitude Plot



First Order LPF TF Phase Plot



First Order HPF Transfer Function

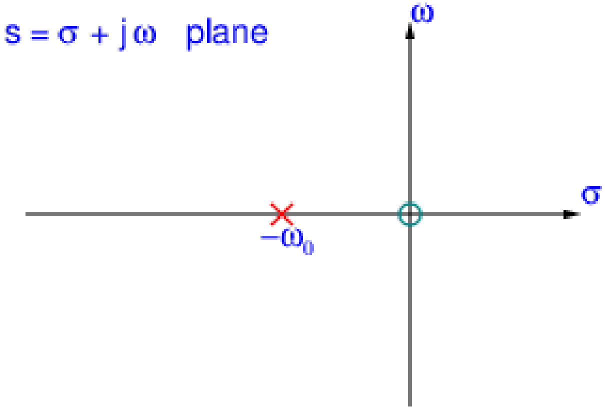
$$T(s) = \frac{s}{s + \omega_0}$$

$$T(j\omega) = \frac{1}{1 - j\omega_0/\omega}$$

$$|T(j\omega)| = \frac{1}{\sqrt{1 + (\omega_0/\omega)^2}}$$

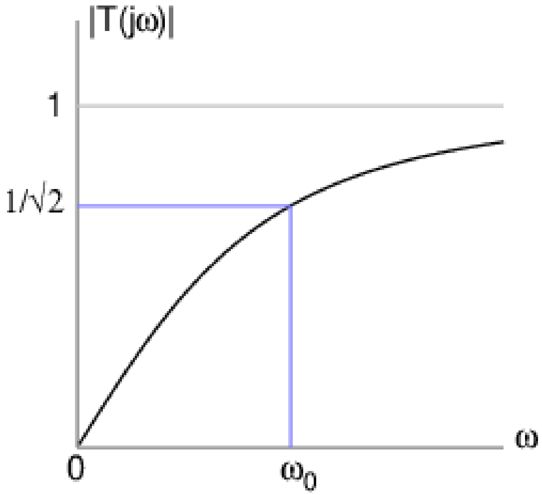
So $|T(j\omega_0)| = 1/\sqrt{2}$.
For $|\omega| \ll \omega_0$, $|T(j\omega)| \approx |\omega|/\omega_0$.

First Order HPF Pole-zero Diagram

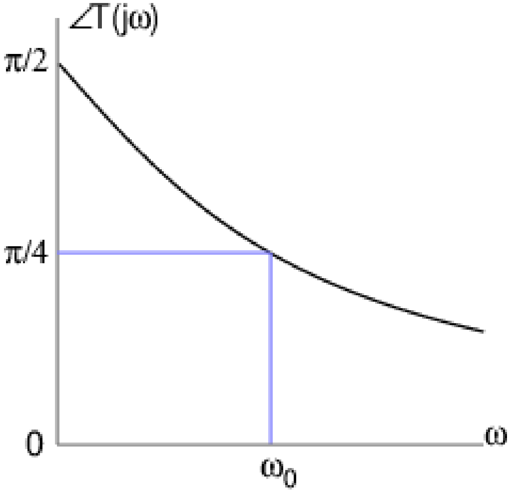


Has one pole and one zero.

First Order HPF TF Magnitude Plot



First Order HPF TF Phase Plot



Second-order Transfer Functions: LPF, BPF, and HPF

Now we recall the second-order transfer functions connected with the spring-mass-dashpot system.
LPF (Lowpass Filter):

$$T_{\mathrm{LPF}}(s) = \frac{\omega_0^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}. \tag{18}$$

BPF (Bandpass Filter):

$$T_{\mathrm{BPF}}(s) = \frac{\frac{\omega_0}{Q}s}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}. \tag{19}$$

HPF (Highpass Filter):

$$T_{\mathrm{HPF}}(s) = \frac{s^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}. \tag{20}$$

When discussing a particular type of filter, the subscript of T may be omitted.

The Second-order Bandpass Transfer Function

$$T(s) = \frac{2\alpha s}{s^2 + 2\alpha s + \omega_0^2}$$

For small loss, that is for small b, or for small α, $T(s)$ has poles at $-\alpha \pm j\sqrt{\omega_0^2 - \alpha^2}$.
So α is the decay constant.
ω_0 is the angular frequency of oscillations for no loss.

Magnitude Response in the Frequency Domain

$$T(s) = \frac{2\alpha s}{s^2 + 2\alpha s + \omega_0^2}$$

$$T(j\omega) = \frac{j2\alpha\omega}{-\omega^2 + j2\alpha\omega + \omega_0^2} = \frac{1}{1 + \frac{\omega_0^2 - \omega^2}{j2\alpha\omega}} = \frac{1}{1 + j\frac{\omega^2 - \omega_0^2}{2\alpha\omega}}$$

Centre Angular Frequency

$$T(j\omega) = \frac{1}{1 + j\frac{\omega^2 - \omega_0^2}{2\alpha\omega}}$$

When is $|T(j\omega)| = 1$?
This happens when $\omega = \pm\omega_0$.
At other values of ω, $|T(j\omega)| < 1$.

Half-power Angular Frequencies

$$T(j\omega) = \frac{1}{1 + j\frac{\omega^2 - \omega_0^2}{2\alpha\omega}}$$

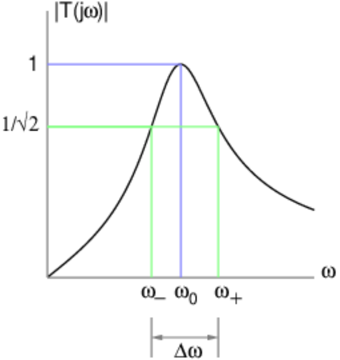
When is $|T(j\omega)| = \frac{1}{\sqrt{2}}$?

This happens when $\frac{\omega^2 - \omega_0^2}{2\alpha\omega} = \pm 1$.
Or, $\omega^2 - \omega_0^2 = \pm 2\alpha\omega$.
The two quadratic equations are,
$\omega^2 - 2\alpha\omega - \omega_0^2 = 0$,
and
$\omega^2 + 2\alpha\omega - \omega_0^2 = 0$.

The positive root of the first quadratic equation is $\omega_+ = \alpha + \sqrt{\alpha^2 + \omega_0^2}$.

The positive root of the second quadratic equation is $\omega_- = -\alpha + \sqrt{\alpha^2 + \omega_0^2}$.

Magnitude Plot of the BPF Transfer Function



Note that $\omega_+\omega_- = \omega_0^2$. Half-power angular bandwidth: $\Delta\omega = \omega_+ - \omega_- = 2\alpha$.
Quality factor

$$Q = \frac{\omega_0}{\Delta\omega} = \frac{\omega_0}{2\alpha}$$

What is Q?

Q is a measure of the selectivity of the BPF. Note that this definition in the frequency domain is the original, exact definition of Q.

Note that $2\alpha = \Delta\omega = \frac{\omega_0}{Q}$.

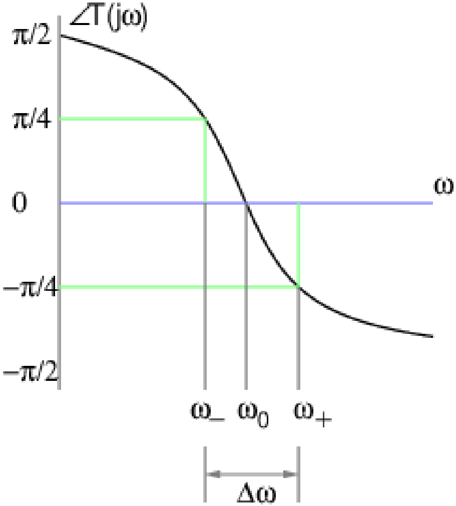
$$\omega_+ = \left(\sqrt{1 + \frac{1}{4Q^2}} + \frac{1}{2Q} \right) \omega_0$$

$$\omega_- = \left(\sqrt{1 + \frac{1}{4Q^2}} - \frac{1}{2Q} \right) \omega_0$$

Remember that ω_0 is the *geometric* mean of ω_+ and ω_-.

It is NOT the arithmetic mean of ω_+ and ω_-.

Phase Plot of the BPF Transfer Function



Phase is easier to measure!

BPF Transfer Function Rewritten

$$T(s) = \frac{2\alpha s}{s^2 + 2\alpha s + \omega_0^2}$$

Since $2\alpha = \frac{\omega_0}{Q}$,

$$T(s) = \frac{\frac{\omega_0}{Q} s}{s^2 + \frac{\omega_0}{Q} s + \omega_0^2}$$

This is the standard form of the transfer function of the BPF.
For the mass-spring-dashpot BPF system, $\omega_0 = \sqrt{k/m}$, and $2\alpha = b/m$. So,

$$Q = \frac{\omega_0}{2\alpha} = \frac{\sqrt{k/m}}{b/m} = \frac{\sqrt{km}}{b}. \tag{21}$$

For other circuits or physical systems, these expressions will need to be determined in terms of the parameters of that system.

General Second Order BPF Transfer Function

$$T(s) = \frac{H\frac{\omega_0}{Q}s}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$$

ω_0: Centre angular frequency
Q: Quality factor
H: Gain factor

Second Order BPF Pole Locations

Find zeros of $s^2 + \frac{\omega_0}{Q}s + \omega_0^2$.

Case $Q > \frac{1}{2}$ (Underdamped)

$$s_1 = -\frac{\omega_0}{2Q} + j\omega_0\sqrt{1 - \frac{1}{4Q^2}}$$

$$s_2 = -\frac{\omega_0}{2Q} - j\omega_0\sqrt{1 - \frac{1}{4Q^2}}$$

Complex conjugate pair of poles. $s_1 s_2 = \omega_0^2$.

Case $Q = \frac{1}{2}$ (Critically damped)

$$s_1 = s_2 = -\omega_0.$$

Equal, negative real poles.

Second Order BPF Pole Locations

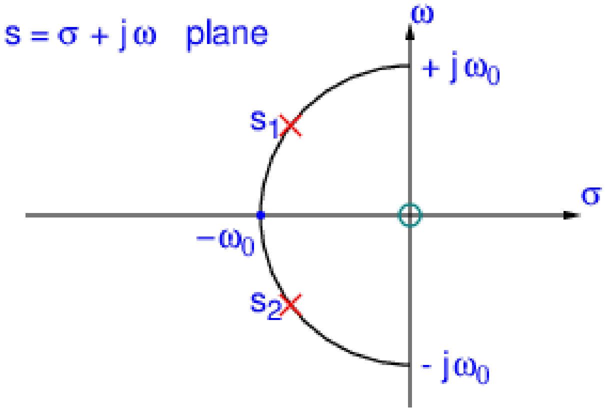
Case $Q < \frac{1}{2}$ (Overdamped)

$$s_1 = -\frac{\omega_0}{2Q} + \omega_0\sqrt{\frac{1}{4Q^2} - 1}$$

$$s_2 = -\frac{\omega_0}{2Q} - \omega_0\sqrt{\frac{1}{4Q^2} - 1}$$

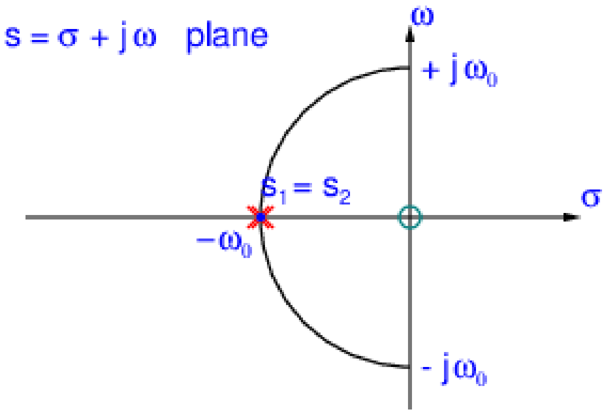
Unequal negative real poles. $s_1 s_2 = \omega_0^2$.

Second Order BPF Pole-zero Diagram (Underdamped)



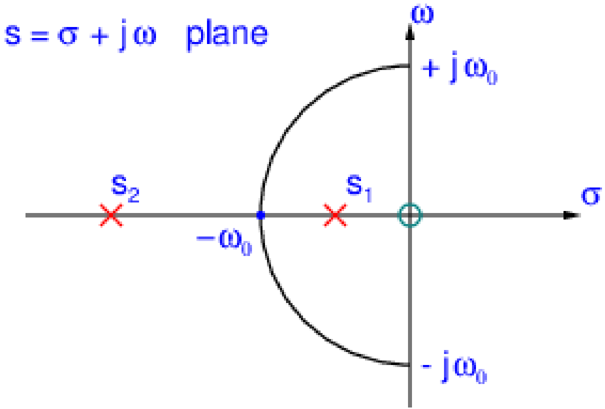
Underdamped system: Shown for $Q > \frac{1}{2}$. Has two poles and one zero.

Second Order BPF Pole-zero Diagram (Critically Damped)



Critically damped system: Shown for $Q = \frac{1}{2}$. Here, $s_1 = s_2 = -\omega_0$. Has two poles and one zero.

Second Order BPF Pole-zero Diagram (Overdamped)



Overdamped system: Shown for $Q < \frac{1}{2}$. Has two poles and one zero.

Second Order LPF and HPF

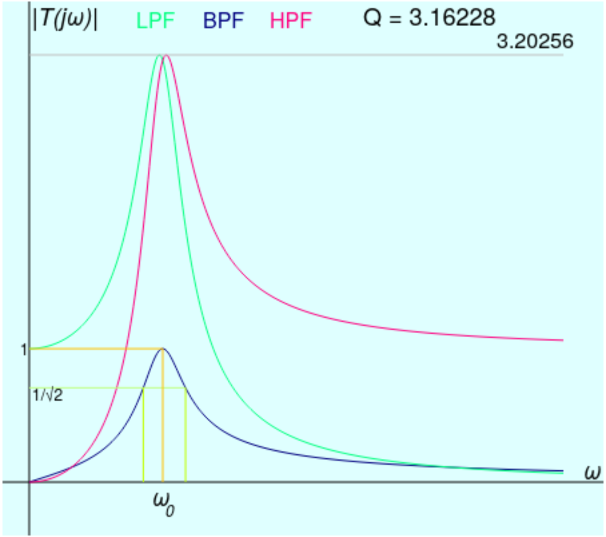
LPF:

$$T_{\mathrm{LPF}}(s) = \frac{\omega_0^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}. \tag{22}$$

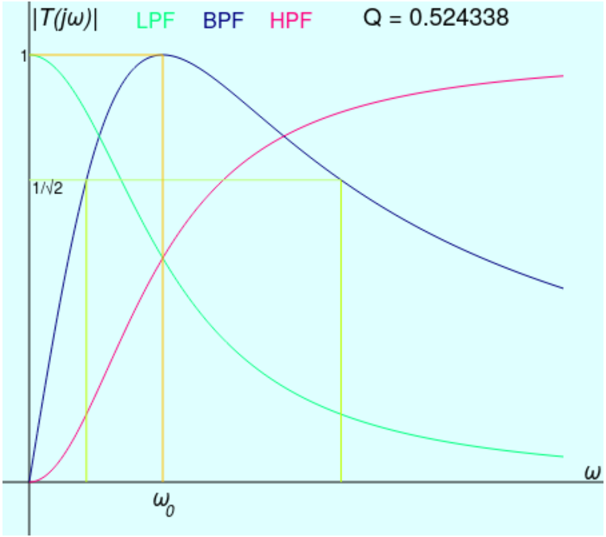
HPF:

$$T_{\mathrm{HPF}}(s) = \frac{s^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}. \tag{23}$$

Case of *Peaking*



Case of *No Peaking*

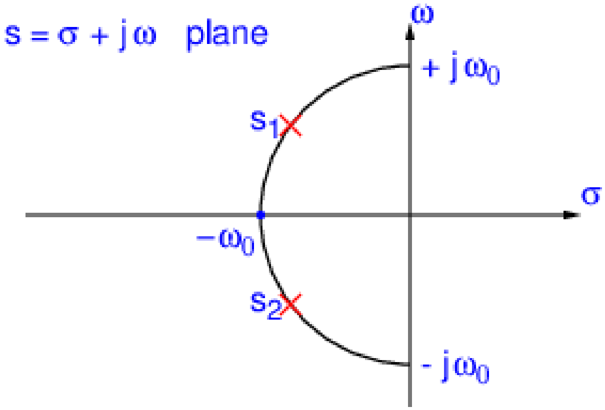


General Second Order LPF Transfer Function

$$T(s) = \frac{H\omega_0^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$$

ω_0: Centre angular frequency
Q: Quality factor
H: Gain factor

Second Order LPF Pole-zero Diagram (Underdamped)



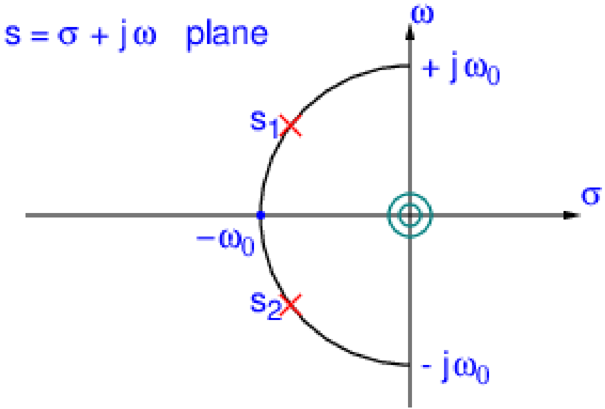
Shown for $Q > \frac{1}{2}$. Has two poles and no zero.

General Second Order HPF Transfer Function

$$T(s) = \frac{Hs^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$$

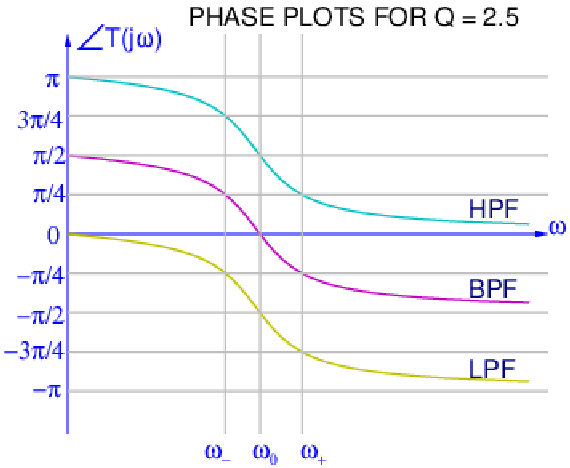
ω_0: Centre angular frequency
Q: Quality factor
H: Gain factor

Second Order HPF Pole-zero Diagram (Underdamped)



Shown for $Q > \frac{1}{2}$. Has two poles and two zeros.

$T(j\omega)$ Phase



Observations

Note that $T_{\mathrm{HPF}}(j\omega)/T_{\mathrm{BPF}}(j\omega) = jQ\omega/\omega_0$, and $T_{\mathrm{LPF}}(j\omega)/T_{\mathrm{BPF}}(j\omega) = -jQ\omega_0/\omega$.
So for positive ω, the HPF phase leads the BPF phase by $\pi/2$, while the LPF phase lags the BPF phase by $\pi/2$, as the plot shows.
In the same way, for the first-order case, HPF phase leads the LPF phase by $\pi/2$.
Points to note:

- Unlike the magnitude plots, the phase plots are monotonic.
- HPF, BPF, and LPF phase plots are very simply related to one another.
- Phase is often easier to measure.

Notation and Terminology

Note that even though the second order LPF and HPF are not really bandpass filters, we still use the notations ω_0 and Q.
The meanings are different, even though the expressions are the same.