IN 221 (AUG) 3:0
Sensors and Transducers
Lecture 5

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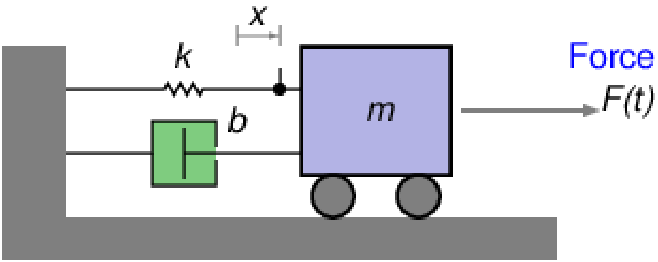
First-order System Example

- In Lecture 3, the spring-mass-dashpot system model was discussed.
- It is a second-order system.
- We have neglected to mention the first-order systems which should be studied first.

Spring-Mass-Dashpot System: Modelling



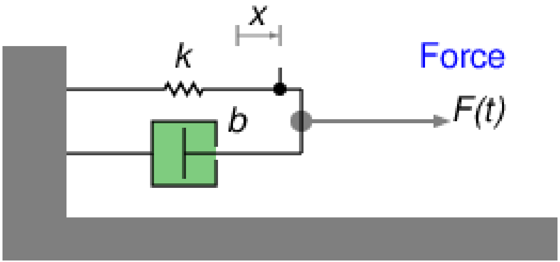
x: Displacement of the mass from its equilibrium position

$$m\ddot{x} + b\dot{x} + kx = F \quad (1)$$

F: Force
What happens when $m \to 0$?

$$b\dot{x} + kx = F \quad (2)$$

Spring-Dashpot System



$$b\dot{x} + kx = F \tag{3}$$

Study of two possible systems:

- Input is F, output is x, or to have the same dimension, $F_{\text{spring}} = kx$, tension force in the spring.
- Input is F, output is $\dot{x}$, or to have the same dimension, $F_{\text{dashpot}} = b\dot{x}$, tension force in the dashpot.

First-order LPF: Input F, output $F_{\text{spring}} = kx$

$$T(s) = \frac{k}{bs + k} = \frac{k/b}{s + k/b} = \frac{\omega_0}{s + \omega_0}, \tag{4}$$

where,

$$\omega_0 = k/b. \tag{5}$$

This is an example of a first-order lowpass filter.

First-order HPF: Input F, output $F_{\text{dashpot}} = b\dot{x}$

$$T(s) = \frac{bs}{bs+k} = \frac{s}{s+k/b} = \frac{s}{s+\omega_0}, \tag{6}$$

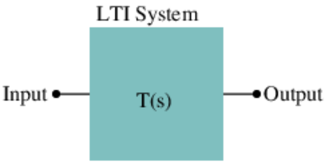
where,

$$\omega_0 = k/b. \tag{7}$$

This is an example of a first-order highpass filter.

Transfer Function: Interpretation

Transfer functions are useful for understanding the behaviour of LTI (linear time-invariant) systems.



An input of e^{st} produces an output of $T(s)e^{st}$ in the steady state, that is after the transients have died down.
This way of thinking is useful even though the formal definition of the transfer functions involves Laplace transforms.

Transfer Function: Properties

- $T(s)$ is real for real s.
- $\overline{T(s)} = T(\overline{s})$.
- For lumped-element systems, that is for systems described by ODEs, $T(s)$ is a ratio of two real polynomials in s.
- Real polynomial: Coefficients are real, even though for complex s its value may be complex.
- Poles and zeros of $T(s)$ are important for the study of the system.

Sinusoidal Input

Let $T(j\omega) = U + jV$, so that $T(-j\omega) = U - jV$.
Input $e^{j\omega t}$ produces output $T(j\omega)e^{j\omega t} =$
$U\cos(\omega t) - V\sin(\omega t) + j\,[U\sin(\omega t) + V\cos(\omega t)]$.
Input $e^{-j\omega t}$ produces output $T(-j\omega)e^{-j\omega t} =$
$U\cos(\omega t) - V\sin(\omega t) - j\,[U\sin(\omega t) + V\cos(\omega t)]$.
Input $\cos(\omega t)$ produces output $U\cos(\omega t) - V\sin(\omega t)$, which is same as

$$\sqrt{U^2 + V^2}\left(\frac{U}{\sqrt{U^2 + V^2}}\cos(\omega t) - \frac{V}{\sqrt{U^2 + V^2}}\sin(\omega t)\right)$$

$$= \sqrt{U^2 + V^2}\cos(\omega t + \Phi) = |T(j\omega)|\cos(\omega t + \Phi)$$

where $\Phi = \arctan(V/U)$, more correctly `atan2(V,U)`, is the angle of $T(j\omega)$.

Meaning of $T(j\omega)$

So for sinusoidal input, the output is also sinusoidal, the amplitude being multiplied by $|T(j\omega)|$, the magnitude of $T(j\omega)$, and the phase being shifted by the angle of $T(j\omega)$.

First-order LPF

$$T(s) = \frac{\omega_0}{s + \omega_0}$$

where, $\omega_0 = k/b$.

First-order HPF

$$T(s) = \frac{s}{s + \omega_0}$$

where, $\omega_0 = k/b$.

First Order LPF Transfer Function

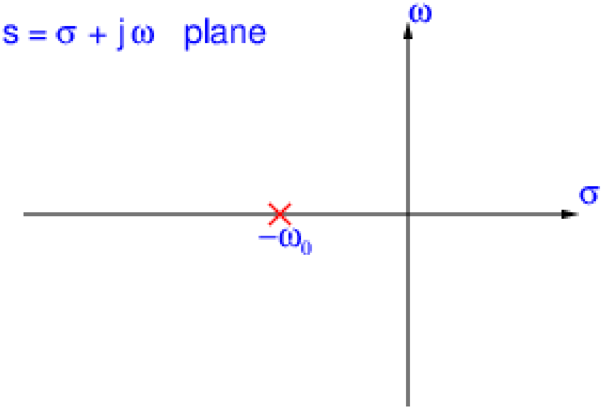
$$T(s) = \frac{\omega_0}{s + \omega_0}$$

$$T(j\omega) = \frac{1}{1 + j\omega/\omega_0}$$

$$|T(j\omega)| = \frac{1}{\sqrt{1 + (\omega/\omega_0)^2}}$$

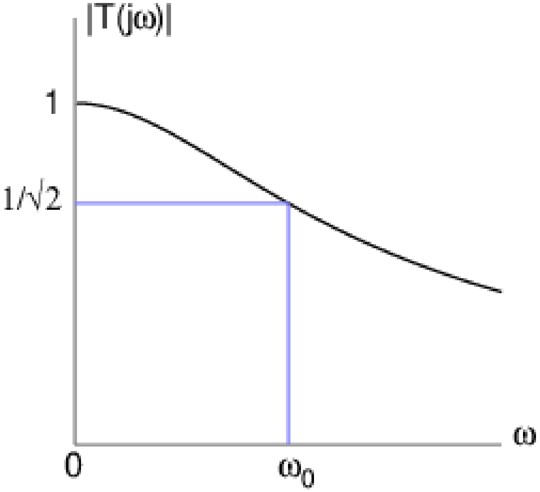
So $|T(j\omega_0)| = 1/\sqrt{2}$.
For $|\omega| \gg \omega_0$, $|T(j\omega)| \approx \omega_0/|\omega|$.

First Order LPF Pole-zero Diagram

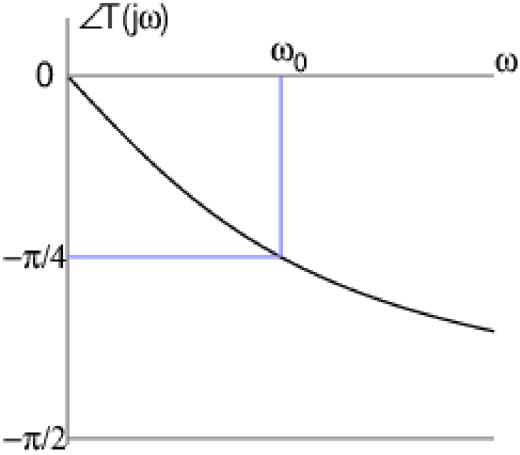


Has one pole and no zero.

First Order LPF TF Magnitude Plot



First Order LPF TF Phase Plot



First Order HPF Transfer Function

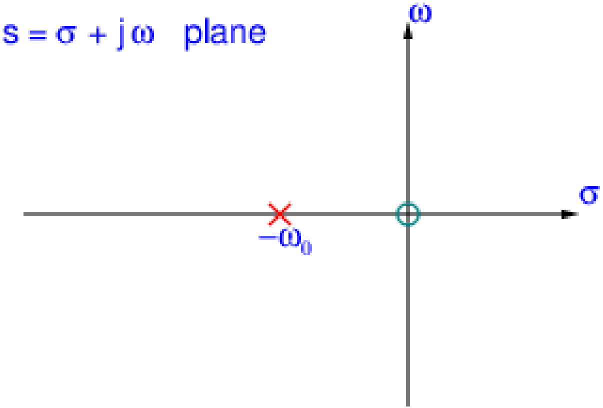
$$T(s) = \frac{s}{s + \omega_0}$$

$$T(j\omega) = \frac{1}{1 - j\omega_0/\omega}$$

$$|T(j\omega)| = \frac{1}{\sqrt{1 + (\omega_0/\omega)^2}}$$

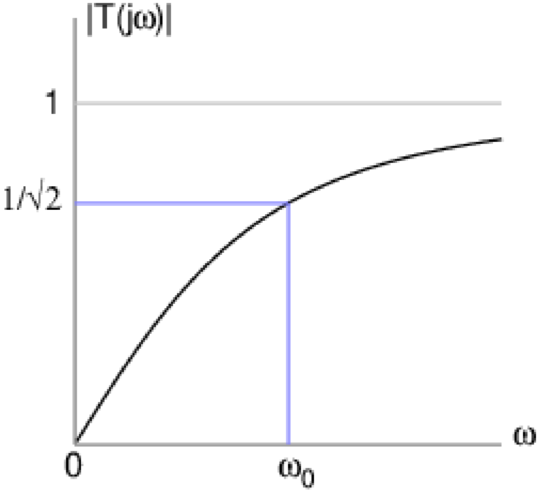
So $|T(j\omega_0)| = 1/\sqrt{2}$.
For $|\omega| \ll \omega_0$, $|T(j\omega)| \approx |\omega|/\omega_0$.

First Order HPF Pole-zero Diagram

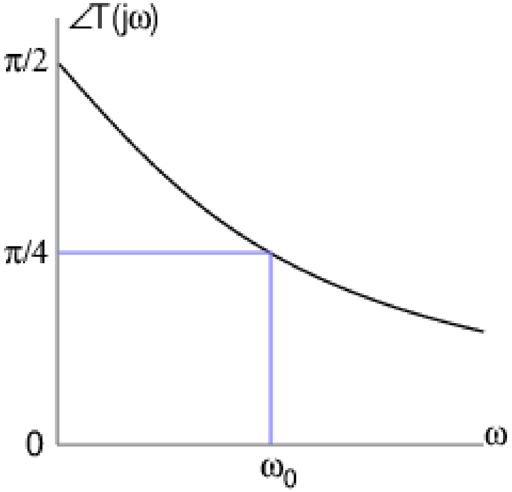


Has one pole and one zero.

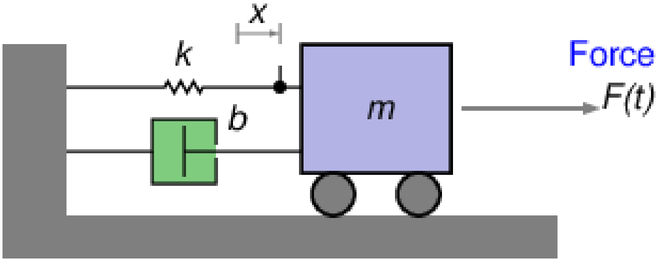
First Order HPF TF Magnitude Plot



First Order HPF TF Phase Plot



Spring-Mass-Dashpot System: Modelling



x: Displacement of the mass from its equilibrium position

$$m\ddot{x} + b\dot{x} + kx = F \tag{8}$$

F: Force
$v = \dot{x}$: Velocity
Relationship between force and velocity:

$$m\ddot{v} + b\dot{v} + kv = \dot{F} \tag{9}$$

Tension in the Dashpot

- Here the applied force $F(t)$ is the input.
- We could consider the velocity $v(t)$ as the output.
- A better choice is to consider the tension in the dashpot, $F_d(t) = bv(t)$, as the output.
- $F_d(t)$ is the force endured by the dashpot.
- Having both input and output as forces makes the mathematics neater.

Relationship between $F(t)$ and $F_d(t)$:

$$m\ddot{F}_d + b\dot{F}_d + kF_d = b\dot{F} \tag{10}$$

Transfer Function

The transfer function is

$$T(s) = \frac{\mathcal{F}_d(s)}{\mathcal{F}(s)} = \frac{bs}{ms^2 + bs + k} = \frac{(b/m)s}{s^2 + (b/m)s + k/m} \tag{11}$$

Or,

$$T(s) = T_{\mathrm{BPF}}(s) = \frac{2\alpha s}{s^2 + 2\alpha s + \omega_0^2},$$

where,

$$\omega_0 = \sqrt{\frac{k}{m}},$$

and

$$b/m = 2\alpha.$$

This transfer function is called the transfer function of a *second-order bandpass filter*, or **BPF**.

Terminology

ω_0 is the angular frequency of oscillations in the absence of damping.
α is called the decay constant.
Both ω_0 and α have dimensions of the inverse of time.
Alternate Notation: ω_n for ω_0, $2\zeta\omega_n$ for 2α
See for example, Section 3.5 of *Linear Control System Analysis and Design with MATLAB* by D'Azzo, Houpis and Sheldon.

Other types of transfer functions in this system

If the force in the spring is taken as the output, then we would get a transfer function of the form

$$T(s) = T_{\mathrm{LPF}}(s) = \frac{\omega_0^2}{s^2 + 2\alpha s + \omega_0^2}.$$

This transfer function is called the transfer function of a *second-order lowpass filter*, or **LPF.**

If the force required to move the mass is taken as the output, then we would get a transfer function of the form

$$T(s) = T_{\mathrm{HPF}}(s) = \frac{s^2}{s^2 + 2\alpha s + \omega_0^2}.$$

This transfer function is called the transfer function of a *second-order highpass filter*, or **HPF.**

BPF $|T(j\omega)|$

For a second-order BPF,

$$T(s) = \frac{2\alpha s}{s^2 + 2\alpha s + \omega_0^2}.$$

So

$$|T(j\omega)| = \left| \frac{2\alpha j\omega}{2\alpha j\omega + \omega_0^2 - \omega^2} \right| = \frac{1}{\sqrt{1 + \left(\frac{\omega_0^2 - \omega^2}{2\alpha\omega}\right)^2}}. \tag{12}$$

Maximum Output: $|T(j\omega)| = 1$ when $\omega = \pm\omega_0$.
ω_0 is called the centre angular frequency.

Sharpness of Response

Half-power Output: This happens when $|T(j\omega)| = 1/\sqrt{2}$.
Or,

$$\frac{\omega_0^2 - \omega^2}{2\alpha\omega} = \pm 1 \tag{13}$$

The two quadratic equations to be solved are

$$\omega^2 - 2\alpha\omega - \omega_0^2 = 0, \tag{14}$$

and

$$\omega^2 + 2\alpha\omega - \omega_0^2 = 0. \tag{15}$$

Half-power Angular Frequencies

The positive root of Eq. 14, called the *upper half-power angular frequency* is

$$\omega_+ = \alpha + \sqrt{\omega_0^2 + \alpha^2} \tag{16}$$

The positive root of Eq. 15, called the *lower half-power angular frequency* is

$$\omega_- = -\alpha + \sqrt{\omega_0^2 + \alpha^2} \tag{17}$$

Note: The negative root of Eq. 14 is $-\omega_-$, and the negative root of Eq. 15 is $-\omega_+$.
$\Delta\omega = \omega_+ - \omega_- = 2\alpha$ is called the half-power bandwidth.
Note that

$$\omega_+\omega_- = \omega_0^2. \tag{18}$$

Quality Factor Q

$$Q = \frac{\omega_0}{\Delta\omega} = \frac{\omega_0}{2\alpha} \tag{19}$$

is a measure of the selectivity or the sharpness of response. A higher Q makes the response more selective.
So

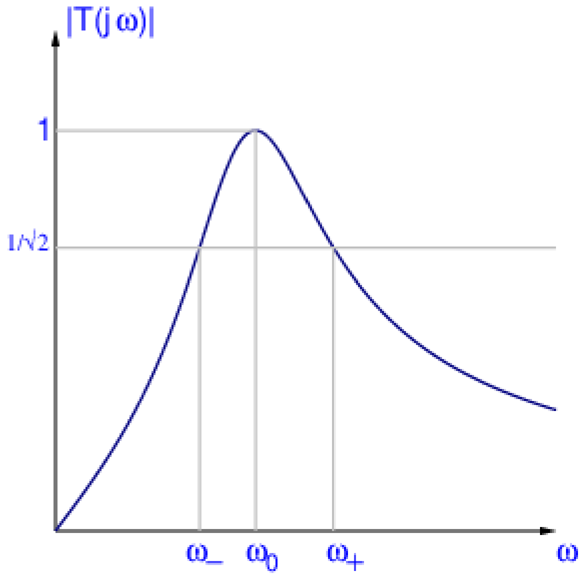
$$2\alpha = \frac{\omega_0}{Q}. \tag{20}$$

In view of this,

$$T(s) = \frac{\frac{\omega_0 s}{Q}}{s^2 + \frac{\omega_0 s}{Q} + \omega_0^2}.$$

Whenever we see a quadratic denominator, we use the Q notation, even if the system is *not* a bandpass system.

Half-power Angular Frequencies Shown for $Q = 1.5$



ω_+ and ω_- in terms of ω_0 and Q

$$\omega_+ = \omega_0 \left(\sqrt{1 + \frac{1}{4Q^2}} + \frac{1}{2Q} \right). \tag{21}$$

$$\omega_- = \omega_0 \left(\sqrt{1 + \frac{1}{4Q^2}} - \frac{1}{2Q} \right). \tag{22}$$

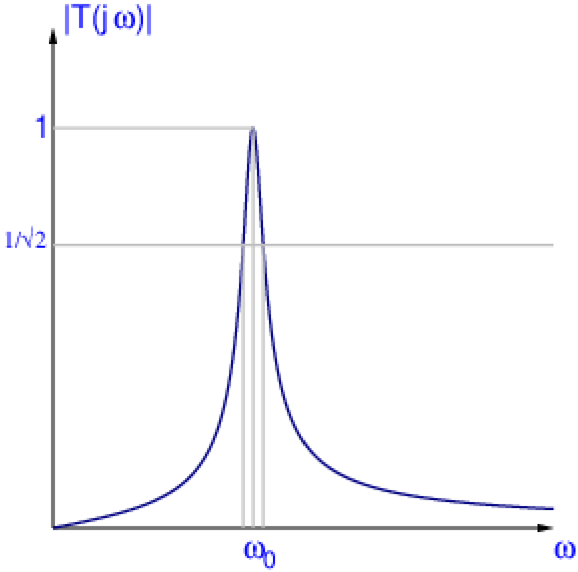
Also, remember that $\omega_+ \omega_- = \omega_0^2$, and $\omega_+ - \omega_- = \omega_0 / Q$.
Note that,

$$\frac{\omega_+}{\omega_0} - \frac{\omega_0}{\omega_+} = \frac{1}{Q}, \tag{23}$$

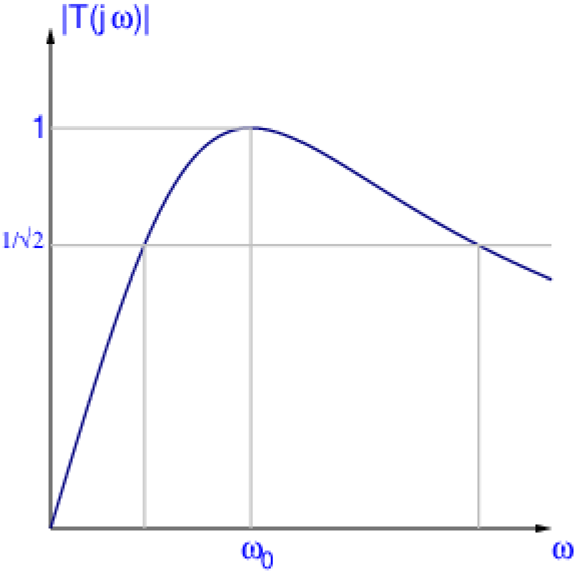
and

$$\frac{\omega_-}{\omega_0} - \frac{\omega_0}{\omega_-} = -\frac{1}{Q}. \tag{24}$$

$|T(j\omega)|$ for $Q = 10$



$|T(j\omega)|$ for $Q = 0.6$



BPF Phase

$$T(j\omega) = \frac{2\alpha j\omega}{2\alpha j\omega + \omega_0^2 - \omega^2} = \frac{j\omega\omega_0/Q}{j\omega\omega_0/Q + \omega_0^2 - \omega^2} = \frac{1}{1 + jQ\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)}. \tag{25}$$

Phase angle is

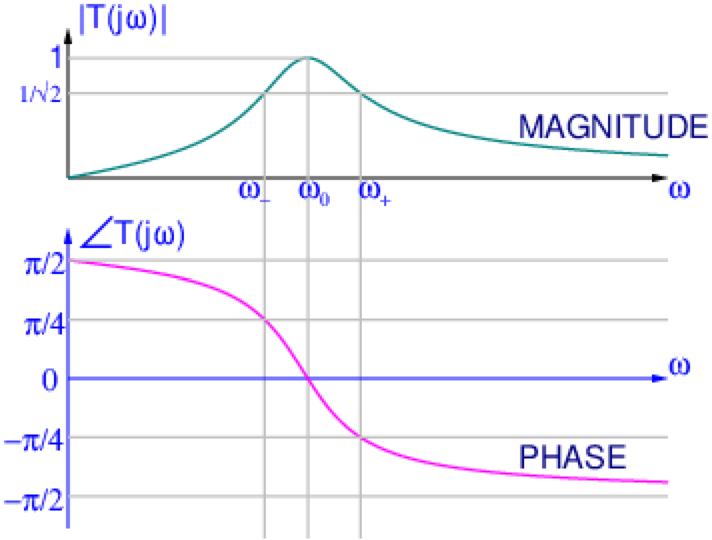
$$\underline{/T(j\omega)} = \arctan\left(Q\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)\right). \tag{26}$$

Special values:

- $\underline{/T(j0)} = \pi/2$.
- $\underline{/T(j\omega_0)} = 0$.
- $\underline{/T(j\infty)} = -\pi/2$.
- $\underline{/T(j\omega_-)} = \pi/4$.
- $\underline{/T(j\omega_+)} = -\pi/4$.

Phase is important because it is often easier to measure.

BPF magnitude and phase on the same plot



BPF MAGNITUDE AND PHASE PLOTS FOR Q = 2.5

Second-order BPF: More general form

We studied a transfer function of the form

$$T(s) = \frac{\frac{\omega_0 s}{Q}}{s^2 + \frac{\omega_0 s}{Q} + \omega_0^2}$$

that occurs in many applications.
The meanings of the Q and ω_0 parameters were understood.
A slightly more general form for the second-order BPF transfer function is

$$T(s) = \frac{H\frac{\omega_0 s}{Q}}{s^2 + \frac{\omega_0 s}{Q} + \omega_0^2}.$$

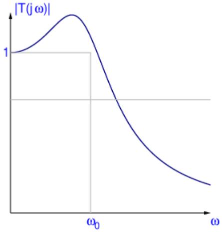
Here H is constant gain or loss factor, useful in systems with amplification or extra losses.

LPF System

Example: MEMS Accelerometer
Input is applied force, output can be the displacement x.
Or, to simplify the mathematics, let the force in the spring, kx, be the output. Then

$$T(s) = \frac{\omega_0^2}{s^2 + \frac{\omega_0 s}{Q} + \omega_0^2}. \tag{27}$$

Even here, the symbol Q is used.



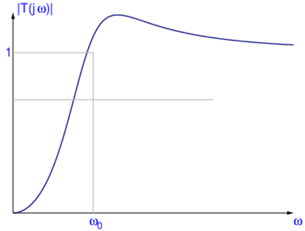
LPF $|T(j\omega)|$ shown for $Q = 1.1$.

HPF System

Example: MEMS Accelerometer
Input is applied force, output can be the acceleration $\ddot{x}$.
Or, to simplify the mathematics, let the force acting on the mass, $m\ddot{x}$, be the output. Then

$$T(s) = \frac{s^2}{s^2 + \frac{\omega_0 s}{Q} + \omega_0^2}. \tag{28}$$

The same symbol Q is used.



LPF $|T(j\omega)|$ shown for $Q = 1.1$.

Second-order Transfer Functions: LPF, BPF, and HPF

Now we recall the second-order transfer functions connected with the spring-mass-dashpot system.
LPF (Lowpass Filter):

$$T_{\mathrm{LPF}}(s) = \frac{\omega_0^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}. \tag{29}$$

BPF (Bandpass Filter):

$$T_{\mathrm{BPF}}(s) = \frac{\frac{\omega_0}{Q}s}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}. \tag{30}$$

HPF (Highpass Filter):

$$T_{\mathrm{HPF}}(s) = \frac{s^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}. \tag{31}$$

When discussing a particular type of filter, the subscript of T may be omitted.

General Second Order BPF Transfer Function

$$T(s) = \frac{H\frac{\omega_0}{Q}s}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$$

ω_0: Centre angular frequency
Q: Quality factor
H: Gain factor

Second Order BPF Pole Locations

Find zeros of $s^2 + \frac{\omega_0}{Q}s + \omega_0^2$.

Case $Q > \frac{1}{2}$ (Underdamped)

$$s_1 = -\frac{\omega_0}{2Q} + j\omega_0\sqrt{1 - \frac{1}{4Q^2}}$$

$$s_2 = -\frac{\omega_0}{2Q} - j\omega_0\sqrt{1 - \frac{1}{4Q^2}}$$

Complex conjugate pair of poles. $s_1 s_2 = \omega_0^2$.

Case $Q = \frac{1}{2}$ (Critically damped)

$$s_1 = s_2 = -\omega_0.$$

Equal, negative real poles.

Second Order BPF Pole Locations

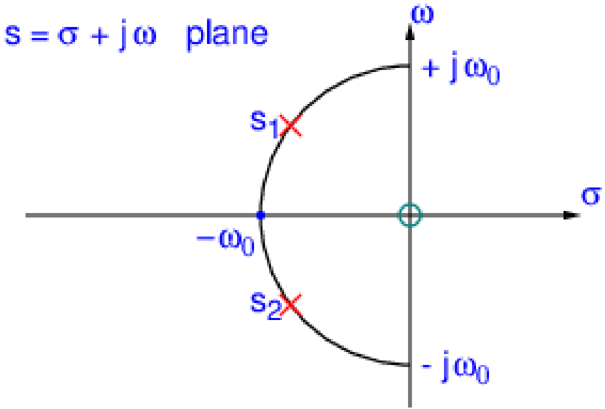
Case $Q < \frac{1}{2}$ (Overdamped)

$$s_1 = -\frac{\omega_0}{2Q} + \omega_0\sqrt{\frac{1}{4Q^2} - 1}$$

$$s_2 = -\frac{\omega_0}{2Q} - \omega_0\sqrt{\frac{1}{4Q^2} - 1}$$

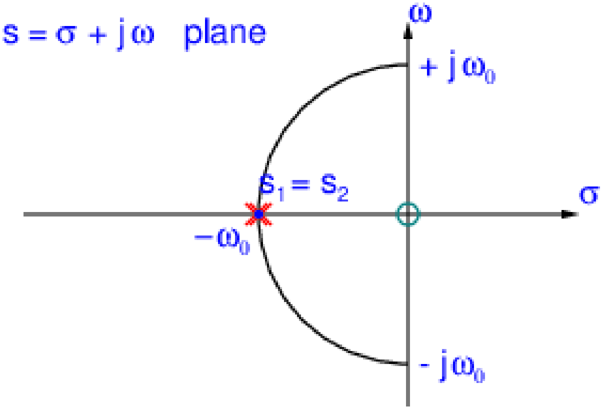
Unequal negative real poles. $s_1 s_2 = \omega_0^2$.

Second Order BPF Pole-zero Diagram (Underdamped)



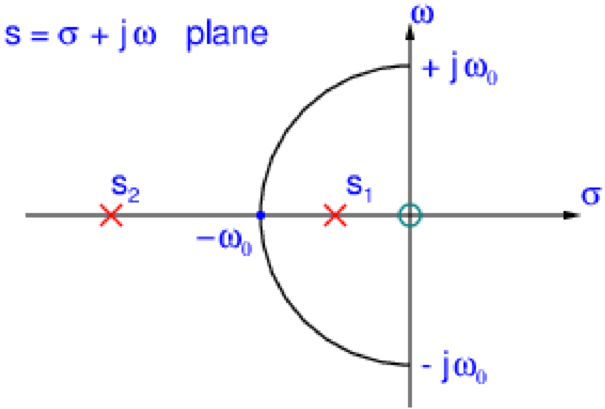
Underdamped system: Shown for $Q > \frac{1}{2}$. Has two poles and one zero.

Second Order BPF Pole-zero Diagram (Critically Damped)



Critically damped system: Shown for $Q = \frac{1}{2}$. Here, $s_1 = s_2 = -\omega_0$. Has two poles and one zero.

Second Order BPF Pole-zero Diagram (Overdamped)



Overdamped system: Shown for $Q < \frac{1}{2}$. Has two poles and one zero.

Second Order LPF and HPF

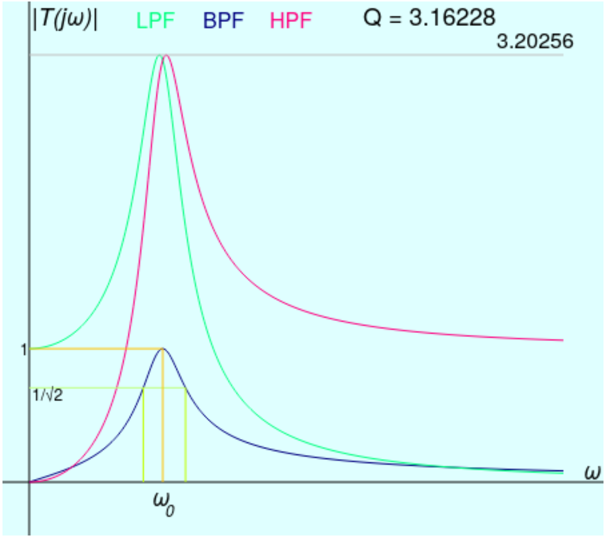
LPF:

$$T_{\mathrm{LPF}}(s) = \frac{\omega_0^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}. \tag{32}$$

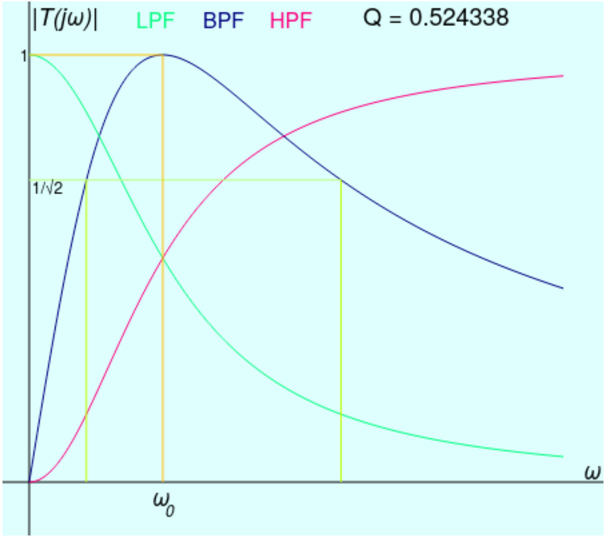
HPF:

$$T_{\mathrm{HPF}}(s) = \frac{s^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}. \tag{33}$$

Case of *Peaking*



Case of *No Peaking*

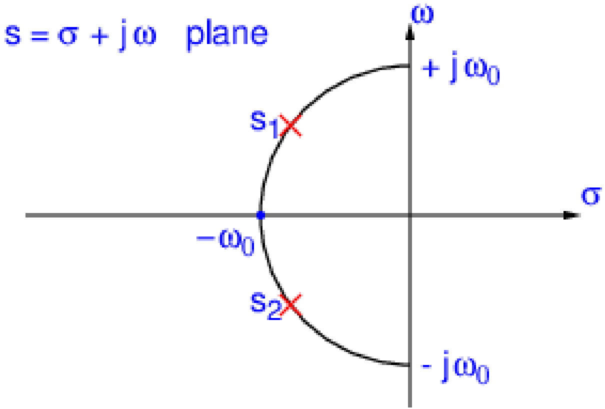


General Second Order LPF Transfer Function

$$T(s) = \frac{H\omega_0^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$$

ω_0: Centre angular frequency
Q: Quality factor
H: Gain factor

Second Order LPF Pole-zero Diagram (Underdamped)



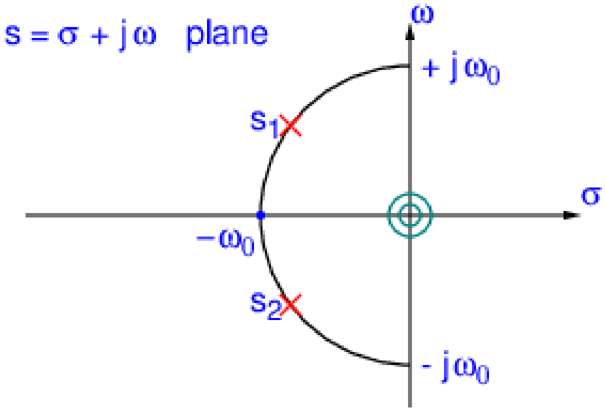
Shown for $Q > \frac{1}{2}$. Has two poles and no zero.

General Second Order HPF Transfer Function

$$T(s) = \frac{Hs^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$$

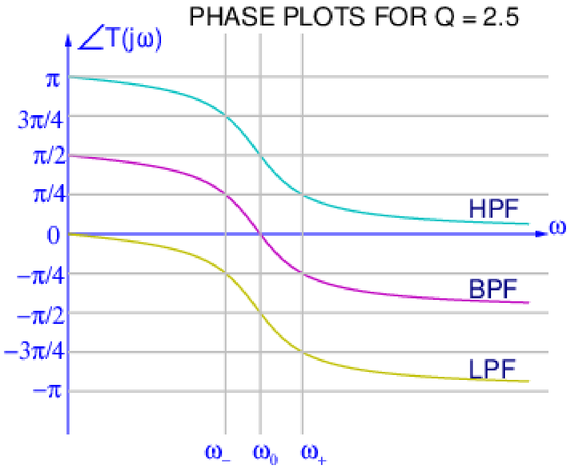
ω_0: Centre angular frequency
Q: Quality factor
H: Gain factor

Second Order HPF Pole-zero Diagram (Underdamped)



Shown for $Q > \frac{1}{2}$. Has two poles and two zeros.

$T(j\omega)$ Phase



Observations

Note that $T_{\text{HPF}}(j\omega)/T_{\text{BPF}}(j\omega) = jQ\omega/\omega_0$, and $T_{\text{LPF}}(j\omega)/T_{\text{BPF}}(j\omega) = -jQ\omega_0/\omega$.
So for positive ω, the HPF phase leads the BPF phase by $\pi/2$, while the LPF phase lags the BPF phase by $\pi/2$, as the plot shows.
In the same way, for the first-order case, HPF phase leads the LPF phase by $\pi/2$.
Points to note:

- Unlike the magnitude plots, the phase plots are monotonic.
- HPF, BPF, and LPF phase plots are very simply related to one another.
- Phase is often easier to measure.

Notation and Terminology

Note that even though the second order LPF and HPF are not really bandpass filters, we still use the notations ω_0 and Q.
The meanings are different, even though the expressions are the same.