IN 221 (AUG) 3:0
Sensors and Transducers
Lecture 6

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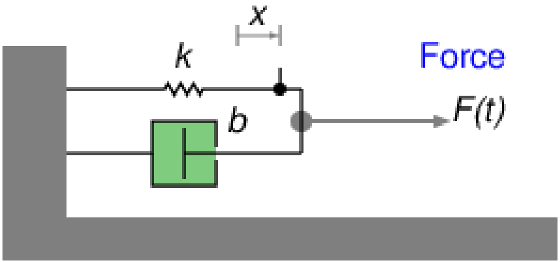
Spring-Dashpot System

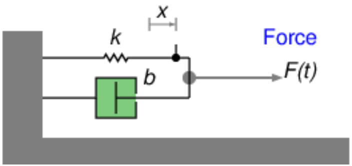


$$b\dot{x} + kx = F \qquad (1)$$

Study of two possible systems:

- Input is F, output is x, or to have the same dimension, $F_{\text{spring}} = kx$, tension force in the spring.
- Input is F, output is $\dot{x}$, or to have the same dimension, $F_{\text{dashpot}} = b\dot{x}$, tension force in the dashpot.

Differential Equation



Symbols introduced for conciseness:

$\Phi = F_{\text{spring}} = kx$,

$\Psi = F_{\text{dashpot}} = b\dot{x} = \frac{b}{k}k\dot{x} = \frac{b}{k}\dot{\Phi}$.

In terms of Φ, the ODE $b\dot{x} + kx = F$ can be expressed as

$$\frac{b}{k}\dot{\Phi} + \Phi = F.$$

Or,

$$\frac{d\Phi}{dt} + \kappa\Phi = \kappa F. \tag{2}$$

Here $\kappa = k/b$ has the dimension of the inverse of time.

Other names for κ

$\kappa = k/b$.
Depending on the situation, it can act in different ways.

- Decay constant: α
- Angular velocity: ω_0

Solution

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} + \kappa\Phi = \kappa F.$$

Multiply both sides by $e^{\kappa t}$. Then the left hand becomes the derivative of $e^{\kappa t}\Phi$.

$$\frac{\mathrm{d}\left(e^{\kappa t}\Phi\right)}{\mathrm{d}t} = \kappa F e^{\kappa t}.$$

Now we integrate both sides from 0 to t.

$$e^{\kappa t}\Phi(t) - \Phi(0) = \kappa \int_0^t F(\tau) e^{\kappa\tau} \, \mathrm{d}\tau.$$

Rearranging terms we get

$$\Phi(t) = \Phi(0)e^{-\kappa t} + e^{-\kappa t}\kappa \int_0^t F(\tau) e^{\kappa\tau} \, \mathrm{d}\tau. \tag{3}$$

Solution for $F(t) = 0$

Case $F(t) = 0$:

$$\Phi(t) = \Phi(0)e^{-\kappa t}. \tag{4}$$

Here we have exponential decay of the solution.
This behaviour can be observed with a light door that has a spring-damper attached to it.
It is to be noted that in this case the tension in the dashpot is

$$\Psi(t) = \frac{1}{\kappa}\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -\Phi(0)e^{-\kappa t} = -\Phi(t). \tag{5}$$

The sum of the tensions in the spring and the dashpot is zero as it should be.
The dashpot is in compression during the return stretched spring to its rest position.

Solution for $F(t) = F_0 e^{st}$

The form e^{st} is special because its derivative is s times itself.
Case $F(t) = F_0 e^{st}$:

$$
\begin{aligned}
\Phi(t) &= \Phi(0)e^{-\kappa t} + e^{-\kappa t}\kappa \int_0^t F(\tau)e^{\kappa\tau}\,\mathrm{d}\tau \\
&= \Phi(0)e^{-\kappa t} + e^{-\kappa t}\kappa \int_0^t F_0 e^{s\tau} e^{\kappa\tau}\,\mathrm{d}\tau \\
&= \Phi(0)e^{-\kappa t} + F_0 e^{-\kappa t}\left(e^{(s+\kappa)t} - 1\right)\frac{\kappa}{s+\kappa} \\
&= \left(\Phi(0) - F_0\frac{\kappa}{s+\kappa}\right)e^{-\kappa t} + \frac{\kappa}{s+\kappa}F_0 e^{st}.
\end{aligned}
$$

The solution has two parts.

Transient and Steady-state Components

The first part,

$$\Phi_{\text{transient}}(t) = \left(\Phi(0) - F_0 \frac{\kappa}{s+\kappa} \right) e^{-\kappa t},$$

represents a transient decay. This is also called the *natural response*.
The second part is

$$\Phi_{\text{steady-state}}(t) = \frac{\kappa}{s+\kappa} F_0 e^{st} = \frac{\kappa}{s+\kappa} F(t) = T(s)F(t).$$

This part is often called the *forced response*. Here, the *transfer function* $T(s) = \kappa/(s+\kappa)$.
$T(s)$ can be other functions of s for other kinds of systems.

Solution for $F(t) = F_0 \cos(\omega t)$

Sinusoidal excitation: $F(t) = F_0 \cos(\omega t)$

$$
\begin{aligned}
\Phi(t) &= \Phi(0)e^{-\kappa t} + e^{-\kappa t}\kappa \int_0^t F(\tau)e^{\kappa\tau}\,\mathrm{d}\tau \\
&= \Phi(0)e^{-\kappa t} + e^{-\kappa t}\kappa \int_0^t F_0\cos(\omega\tau)e^{\kappa\tau}\,\mathrm{d}\tau \\
&= \left(\Phi(0) - F_0\frac{1}{1+(\omega/\kappa)^2}\right)e^{-\kappa t} + F_0\frac{\cos(\omega t) + (\omega/\kappa)\sin(\omega t)}{1+(\omega/\kappa)^2}.
\end{aligned}
$$

This solution was obtained by directly using the formula for $\int \cos(\omega\tau)e^{\kappa\tau}\,\mathrm{d}\tau$.
But it could also have been obtained by using the result for $F(t) = F_0 e^{st}$.

Solution obtained by superposition

$$F(t) = F_0 \cos(\omega t) = \frac{1}{2}F_0 e^{j\omega t} + \frac{1}{2}F_0 e^{-j\omega t}$$

The steady-state solution for input $\frac{1}{2}F_0 e^{j\omega t}$ is $\frac{\kappa}{j\omega+\kappa}\frac{1}{2}F_0 e^{j\omega t} = \frac{1}{2}F_0 \frac{\cos(\omega t)+j\sin(\omega t)}{1+j\omega/\kappa}$.
The steady-state solution for input $\frac{1}{2}F_0 e^{-j\omega t}$ is $\frac{\kappa}{-j\omega+\kappa}\frac{1}{2}F_0 e^{-j\omega t} = \frac{1}{2}F_0 \frac{\cos(\omega t)-j\sin(\omega t)}{1-j\omega/\kappa}$.
Adding these solutions we obtain

$$F_0 \frac{\cos(\omega t) + (\omega/\kappa)\sin(\omega t)}{1 + (\omega/\kappa)^2}$$

after simplification.
The form ω/κ suggests that κ plays the role of a cut-off angular frequency here.
Let $\kappa = \omega_0 = 2\pi f_0$.
f_0 is an important frequency for this system.
In terms of problem parameters, $f_0 = k/(2\pi b)$.

Plots using MATLAB

Now the expression for the output for simusoidal excitation is converted to a MATLAB function.
Then it is used to generate some plots.

Programme Listing: Part 1 of sd.m

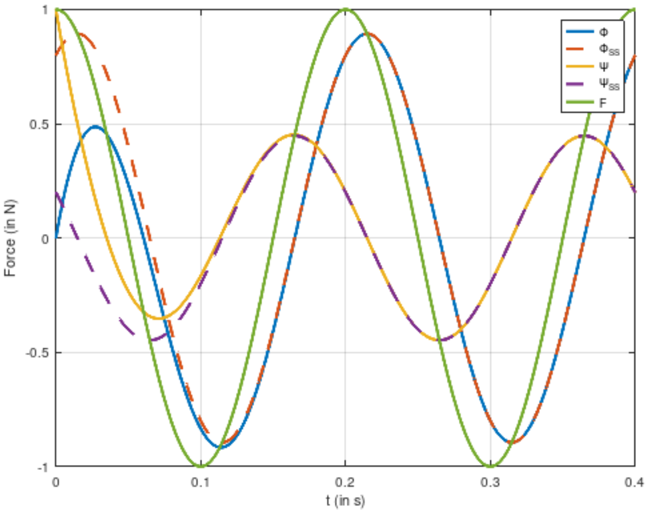
```
function sd(f, f0, tMax, nPts, F0, Phi0)
    omega = 2 * pi * f;
    omega0 = 2 * pi * f0;
    tArr = linspace(0, tMax, nPts);
    cArr = cos(omega * tArr);
    sArr = sin(omega * tArr);
    eArr = exp(-omega * tArr);
    v = f / f0;
    den = 1 + v * v;
    cFac = F0 / den;
    sFac = v * cFac;
    eFac = Phi0 - cFac;
    PhiTrans = eFac * eArr;
    PhiSS = cFac * cArr + sFac * sArr;
    PhiArr = PhiTrans + PhiSS;
```

Programme Listing: Part 2 of sd.m

```
PsiTrans = -PhiTrans;
PsiSS = v * v * cFac * cArr - sFac * sArr;
PsiArr = PsiTrans + PsiSS;
FArr = F0 * cArr;
plot(tArr, PhiArr, 'linewidth', 2, ...
     tArr, PhiSS, 'linewidth', 2, '--', ...
     tArr, PsiArr, 'linewidth', 2, ...
     tArr, PsiSS, 'linewidth', 2, '--', ...
     tArr, FArr, 'linewidth', 2);
legend('\Phi', '\Phi_{SS}', '\Psi', '\Psi_{SS}', 'F');
xlabel('t (in s)');
ylabel('Force (in N)');
grid on;
```

Output for $f = 5\,\text{Hz}$, $f_0 = 10\,\text{Hz}$.

```
octave:1> sd(5, 10, 0.4, 1001, 1, 0)
```

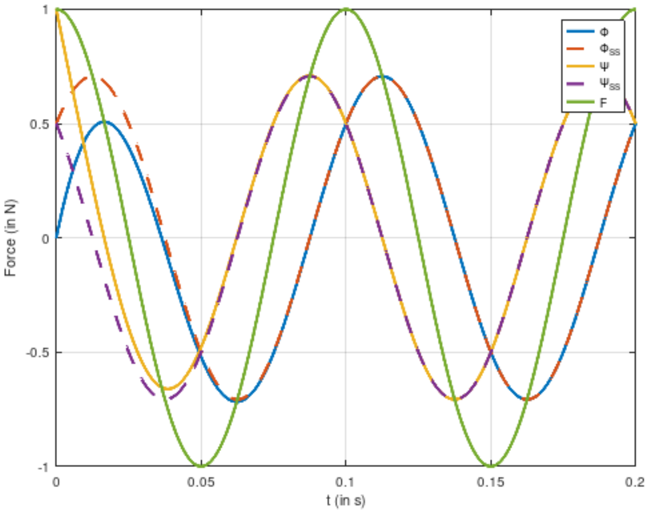


Observations

- As the transients decay, the responses $\Phi(t)$ and $\Psi(t)$, both are very close to sinusoidal.
- The amplitude of $\Phi(t)$ is larger than that of $\Psi(t)$.
- This is because $f < f_0$.
- The phase of $\Phi(t)$ lags, while that of $\Psi(t)$ leads the phase of $F(t)$.

Output for $f = 10\,\text{Hz}$, $f_0 = 10\,\text{Hz}$.

```
octave:1> sd(10, 10, 0.2, 1001, 1, 0)
```

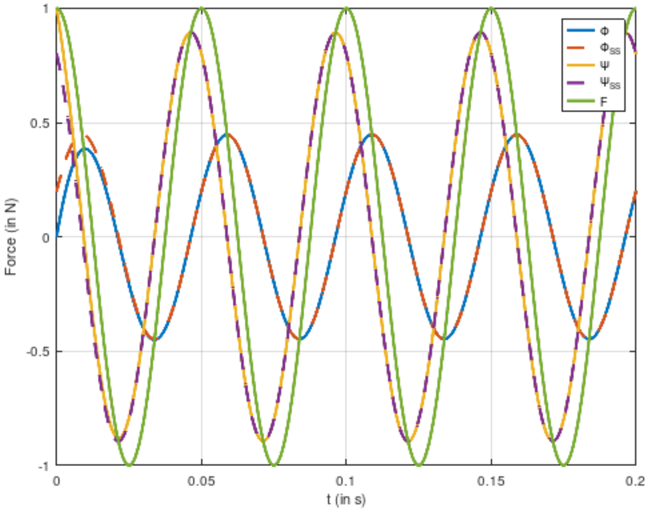


Observations

- As the transients decay, the responses $\Phi(t)$ and $\Psi(t)$, both are very close to sinusoidal.
- The amplitude of $\Phi(t)$ is same as that of $\Psi(t)$.
- The amplitudes of $\Phi(t)$ and $\Psi(t)$ here are $1/\sqrt{2}$ the amplitude of $F(t)$.
- This is because $f = f_0$.
- The phase of $\Phi(t)$ lags, while that of $\Psi(t)$ leads the phase of $F(t)$.
- The phases are both $\pi/4$ or 45 degree in magnitude.

Output for $f = 20$ Hz, $f_0 = 10$ Hz.

```
octave:1> sd(20, 10, 0.2, 1001, 1, 0)
```



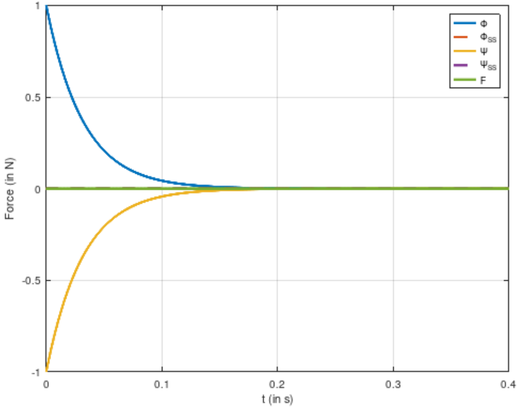
Observations

- As the transients decay, the responses $\Phi(t)$ and $\Psi(t)$, both are very close to sinusoidal.
- The amplitude of $\Phi(t)$ is smaller than that of $\Psi(t)$.
- This is because $f > f_0$.
- The phase of $\Phi(t)$ lags, while that of $\Psi(t)$ leads the phase of $F(t)$.

Transient Response

To look at the transient respose, we make $F_0 = 0$, $\Phi_0 = 1$.

```
octave:1> sd(5, 10, 0.4, 1001, 0, 1)
```



We note that the transient part of $\Psi(t)$ is negative of the transient part of $\Phi(t)$, as it should be.