# The Dehmelt Approximation

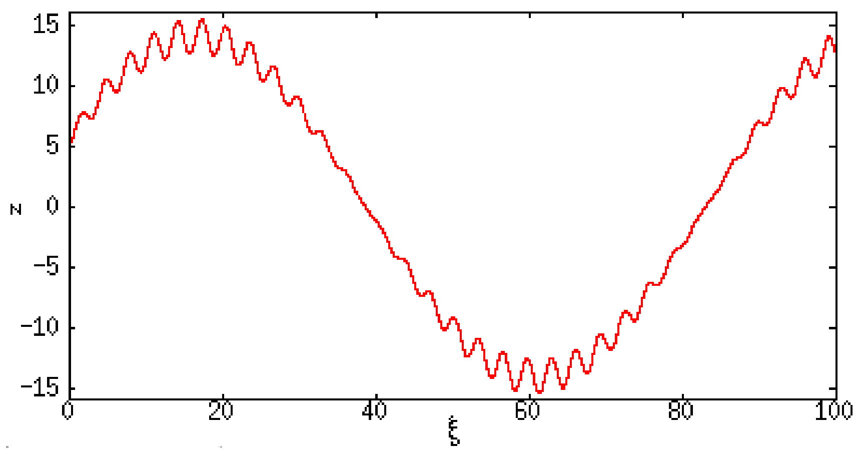
February 4, 2020

## 1 The Dehmelt Approximation

We begin with the Mathieu equation:

$$z'' + [a - 2q\cos(2\xi)]\, z = 0 \tag{1}$$

In this discussion, prime ($'$) denotes differentiation with respect to $\xi$, the normalized time. There is no explicit mention of time here. For small values of $a$ and $q$, the evolution of $z(\xi)$ looks as in the following figure in which $z(0) = 5$, $z'(0) = 1$, $q = 0.2$, $a = -0.015$.




In the figure above, $z(\xi)$ looks like the sum of a slowly oscillating quantity with a large fixed amplitude and rapidly oscillating quantity of a small modulated amplitude.

Even though we have discussed numerical techniques for calculating the solution and other quantities associated with it, for *small* values of $a$ and $q$, there is an approximation due to Dehmelt which provides simple estimates for both the slow and the fast components of the solution. Dehmelt's approximation also provides a great deal of insight on the behaviour of physical systems described by the Mathieu equation. A crude derivation of the Dehmelt approximation will now be given. We write,

$$z = Z + \zeta, \tag{2}$$

where $Z$ is a slowly varying quantity and $\zeta$ is a rapidly varying quantity. We assume that $\zeta \ll Z$, but $\zeta' \gg Z'$ and $\zeta'' \gg Z''$. Averaging over a cycle of $\zeta$ would make $\zeta$ and its derivatives 0, but leave $Z$ and its derivatives intact. Substituting $z$ as $Z + \zeta$ in Eq.(1) we get

$$Z'' + \zeta'' + [a - 2q\cos(2\xi)]\,(Z + \zeta) = 0 \tag{3}$$

First we consider the significant rapidly oscillating parts of Eq.(3): They are $\zeta''$, and $-2q\cos(2\xi)Z$. The part $[a - 2q\cos(2\xi)]\,\zeta$, while rapidly oscillating, is not significant since both $\zeta$, and $a - 2q\cos(2\xi)$ are assumed to be small. Thus we have

$$\zeta'' - 2q\cos(2\xi)Z = 0 \tag{4}$$

Since $Z$ is assumed to be changing slowly, in Eq.(4) it may be considered a constant. Then after two integrations, we have the rapidly oscillating $\zeta$ in terms of $Z$ as:

$$\zeta = -\frac{q}{2}\cos(2\xi)Z \tag{5}$$

Substituting this form of $\zeta$ in Eq.(3) we get

$$Z'' + aZ - \frac{aq}{2}\cos(2\xi)Z + q^2\cos^2(2\xi)Z = 0 \tag{6}$$

Note that the $\zeta''$ term has cancelled the $-2q\cos(2\xi)Z$ term. Averaging Eq.(6) over a cycle of $\zeta$ and noting that the average of a cosine square is 1/2, and that of a cosine is 0, we get Dehmelt's equation for the evolution of $Z$:

$$Z'' + \left(a + \frac{q^2}{2}\right)Z = 0 \tag{7}$$

This solution of this equation is of the form:

$$Z(\xi) = Z(0)\cos\left(\Omega_s\xi\right) + \frac{Z'(0)}{\Omega_s}\sin\left(\Omega_s\xi\right), \tag{8}$$

where,

$$\Omega_s = \sqrt{a + \frac{q^2}{2}}, \tag{9}$$

is Dehmelt's *slow* angular frequency, and $Z(0)$ and $Z'(0)$ are initial value constants which need to be determined from the given values of $z(0)$ and $z'(0)$. Once $Z(\xi)$ is determined from Eq.(8), $\zeta$ can be computed using Eq.(5).

Substituting the expression for $\zeta$ from Eq.(5) in Eq.(2) we get $z$ in terms of $Z$.

$$z = \left[1 - \frac{q}{2}\cos(2\xi)\right]Z \tag{10}$$

To get $Z(0)$ from $z(0)$ we use Eq.(10) with $\xi = 0$:

$$Z(0) = \frac{z(0)}{1 - \frac{q}{2}} \tag{11}$$

It is not clear how legitimate differentiating approximations such as Dehmelt's is. However, if we ignore such concerns and differentiate Eq.(10) with respect to $\xi$ we get,

$$z' = \left[1 - \frac{q}{2}\cos(2\xi)\right]Z' + q\sin(2\xi)Z. \tag{12}$$

Then setting $\xi = 0$ in Eq.(12) we can express $Z'(0)$ as

$$Z'(0) = \frac{z'(0)}{1 - \frac{q}{2}} \tag{13}$$

Now we should look at the web page, `./compareDehmelt.html`, which compares Dehmelt's approximation with the actual numerical solution.

In a physical problem, in which $f$ is the drive frequency, $\Omega_s f/2$ would be Dehmelt's approximation to the *secular frequency*, which is the frequency of the slow oscillations.