IN 277 Notes 2
Some Examples and Introduction to Active Filters

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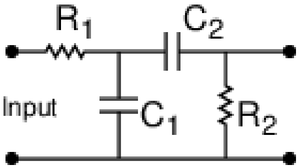
Filters

- Used for selecting some frequencies and rejecting others.
- Some common uses:
  - Reducing noise
  - Frequency division multiplexing
  - Enhancing one harmonic of a periodic signal
- Our plan:
  1. Study simple filters or building blocks
  2. Combine these building blocks to make more complex filters

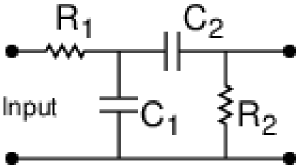
Example 1: RCCR BPF



Recall from Notes 1 that
$A = sR_1C_1 + 1 + \frac{R_1}{R_2} + \frac{R_1C_1}{R_2C_2} + \frac{1}{sR_2C_2}$.
So

$$T(s) = \frac{1}{A} = \frac{\frac{1}{R_1C_1}s}{s^2 + \frac{1+\frac{R_1}{R_2}+\frac{R_1C_1}{R_2C_2}}{R_1C_1}s + \frac{1}{R_1R_2C_1C_2}}$$

Example 1: RCCR BPF $T(s)$ in Standard Form



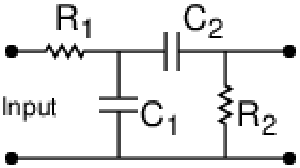
Comparing with the standard form we see that

$$\omega_0 = \frac{1}{\sqrt{R_1 R_2 C_1 C_2}}$$

$$Q = \frac{\sqrt{\frac{R_1 C_1}{R_2 C_2}}}{1 + \frac{R_1}{R_2} + \frac{R_1 C_1}{R_2 C_2}}$$

$$H = \frac{1}{1 + \frac{R_1}{R_2} + \frac{R_1 C_1}{R_2 C_2}}$$

Example 1: Special Case



If $R_1 = R_2 = R$, and $C_1 = C_2 = C$, we have

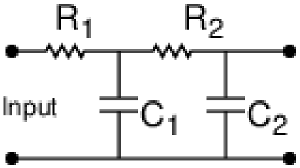
$$\omega_0 = \frac{1}{RC}$$

$$Q = \frac{1}{3}$$

$$H = \frac{1}{3}$$

Not very selective at all!

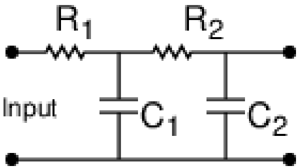
Example 2: RCRC LPF



Compute the ABCD matrix of the network to show that
$A = s^2 R_1 R_2 C_1 C_2 + (R_1 C_1 + R_1 C_2 + R_2 C_2)s + 1$.
So

$$T(s) = \frac{1}{A} = \frac{\frac{1}{R_1 R_2 C_1 C_2}}{s^2 + \frac{R_1 C_1 + R_1 C_2 + R_2 C_2}{R_1 R_2 C_1 C_2} s + \frac{1}{R_1 R_2 C_1 C_2}}$$

Example 2: RCRC LPF $T(s)$ in Standard Form



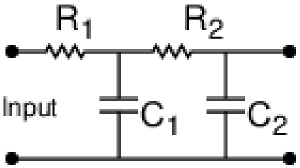
Comparing with the standard form we see that

$$\omega_0 = \frac{1}{\sqrt{R_1 R_2 C_1 C_2}}$$

$$Q = \frac{\sqrt{R_1 R_2 C_1 C_2}}{R_1 C_1 + R_1 C_2 + R_2 C_2}$$

$$H = 1$$

Example 2: Special Case



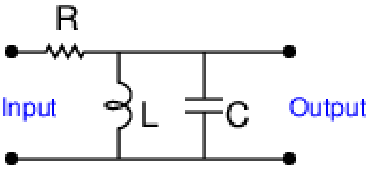
If $R_1 = R_2 = R$, and $C_1 = C_2 = C$, we have

$$\omega_0 = \frac{1}{RC}$$

$$Q = \frac{1}{3}$$

$$H = 1$$

Example 3: BPF from the Parallel RLC Network



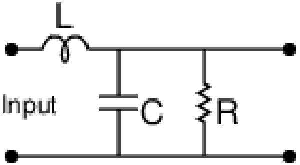
$$T(s) = \frac{\frac{s}{RC}}{s^2 + \frac{s}{RC} + \frac{1}{LC}}$$

Comparing with the standard form we see that $\omega_0 = \frac{1}{\sqrt{LC}}$, $Q = \frac{R}{\sqrt{L/C}}$, and $H = 1$.

Note that the expression for Q here differs from the expression that was derived for the BPF based on the series RLC circuit.

Here Q is proportional to R, there it was inversely proportional to R.

Example 4: LPF from the Parallel RLC Network

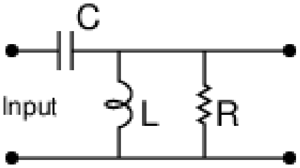


$$T(s) = \frac{\frac{1}{LC}}{s^2 + \frac{s}{RC} + \frac{1}{LC}}$$

Comparing with the standard form we see that $\omega_0 = \frac{1}{\sqrt{LC}}$, $Q = \frac{R}{\sqrt{L/C}}$, and $H = 1$.

This circuit is used for impedance matching in industrial applications.

Example 5: HPF from the Parallel RLC Network



$$T(s) = \frac{s^2}{s^2 + \frac{s}{RC} + \frac{1}{LC}}$$

Comparing with the standard form we see that $\omega_0 = \frac{1}{\sqrt{LC}}$, $Q = \frac{R}{\sqrt{L/C}}$, and $H = 1$.

This circuit is also used for impedance matching in industrial applications.

Why RC Filters?

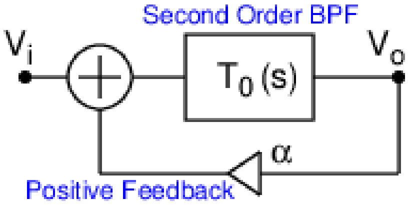
- Inductors are practical at high frequencies.
- At audio frequencies, inductors tend to be bulkier and more expensive compared to resistors and capacitors.
- So there is a desire to make audio frequency filters using resistors and capacitors only.
- But, passive RC second order networks seem to have low Q.
- How can we get more Q?
- The answer is the *active filter*.
- Active filters use amplification to compensate for the losses.

Q-Enhancement using Positive Feedback

Consider a second order BPF whose transfer function is

$$T_0(s) = \frac{H_0 \frac{\omega_{00}}{Q_0} s}{s^2 + \frac{\omega_{00}}{Q_0} s + \omega_{00}^2}$$

The extra 0s in the subscripts are there to indicate original parameters.
Now let us use positive feedback as shown.



What is the new transfer function?

New Transfer Function

$V_o = T_0(s)(V_i + \alpha V_o)$
$V_o(1 - \alpha T_0(s)) = T_0(s) V_i$

$$T(s) = \frac{V_o}{V_i} = \frac{T_0(s)}{1 - \alpha T_0(s)}$$

Substitution of the expression for $T_0(s)$ and simplification gives us

$$T(s) = \frac{H_0 \frac{\omega_{00}}{Q_0} s}{s^2 + (1 - \alpha H_0) \frac{\omega_{00}}{Q_0} s + \omega_{00}^2}$$

New Parameters

Comparing with the standard form

$$T(s) = \frac{H\frac{\omega_0}{Q}s}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$$

we see the following.
$\omega_0 = \omega_{00}$. Centre angular frequency does NOT change.

$$Q = \frac{Q_0}{1 - \alpha H_0}$$

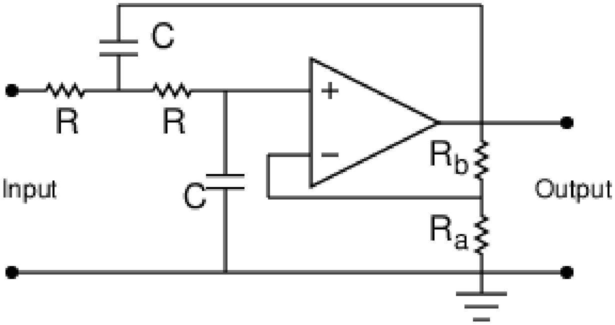
$$H = \frac{H_0}{1 - \alpha H_0}$$

Both Q and H are *enhanced* by the factor $\frac{1}{1-\alpha H_0}$.
We need to be careful. If αH_0 exceeds 1, the circuit will oscillate.

Q-Enhancement: Practical Circuits

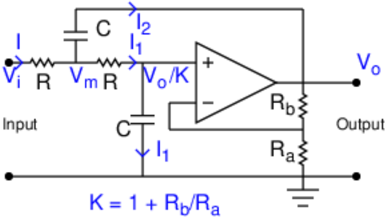
The positive feedback scheme that was described can be implemented using two operational amplifiers.
In practice, only one operational amplifier may be enough.
Not only second order BPF, even second order LPF and HPF circuits can have their Q enhanced using amplifiers.
The next circuit is a practical LPF circuit.

The Sallen-Key Lowpass Filter



DC Gain is $K = 1 + \frac{R_b}{R_a}$.

The Sallen-Key Lowpass Filter: Analysis



We start with the output voltage V_o.

As we have a non-inverting amplifier of gain $K = 1 + R_b/R_a$, the voltage at the non-inverting input of the amplifier is V_o/K.

$I_1 = sC\frac{V_o}{K}$.

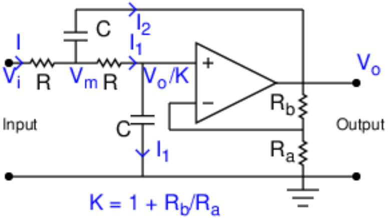
$V_m = \frac{V_o}{K} + I_1 R = (1 + sRC)\frac{V_o}{K}$.

$I_2 = sC(V_m - V_o) = sC(1 - K + sRC)\frac{V_o}{K}$.

$I = I_1 + I_2 = sC(2 - K + sRC)\frac{V_o}{K}$.

$V_i = V_m + RI = \left[1 + (3 - K)sRC + (sRC)^2\right]\frac{V_o}{K}$.

The Sallen-Key LPF: Transfer Function



So

$$T(s) = \frac{V_o}{V_i} = \frac{K}{(sRC)^2 + (3-K)sRC + 1} = \frac{K\frac{1}{(RC)^2}}{s^2 + \frac{3-K}{RC}s + \frac{1}{(RC)^2}}$$

Comparing with the standard forms we see that we have a second order LPF with

$\omega_0 = \frac{1}{RC}$,

$Q = \frac{1}{3-K}$,

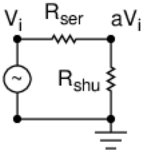
and

$H = K = 1 + R_b/R_a$.

The Sallen-Key LPF: Gain Reduction

The circuit just described has a DC gain of $K = 1 + R_b/R_a$ which exceeds 1.
In many applications, we want a DC gain H_{desired}, which is 1 or less.
This could be achieved by using a voltage divider of division ratio $a = H_{\text{desired}}/K$, followed by a unity gain buffer.
But this can also be achieved by splitting the input resistor into two parts.

Split Input Resistor



We require that the voltage division ratio should be

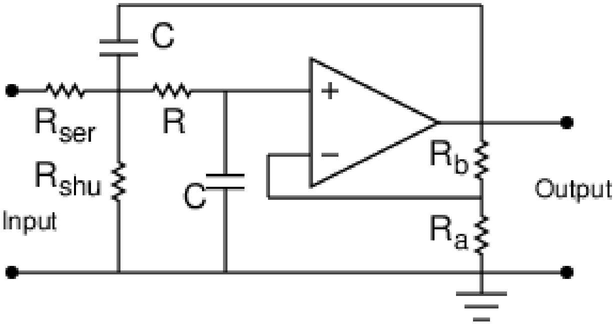
$$\frac{R_{\text{shu}}}{R_{\text{ser}} + R_{\text{shu}}} = a,$$

and the parallel combination of R_{shu} and R_{ser} should be R. So

$$\frac{R_{\text{ser}} R_{\text{shu}}}{R_{\text{ser}} + R_{\text{shu}}} = R.$$

Dividing the second equation by the first, we get
$R_{\text{ser}} = R/a$.
Then solving for R_{shu} we get $R_{\text{shu}} = R/(1 - a)$.

The Sallen-Key LPF: Practical Circuit

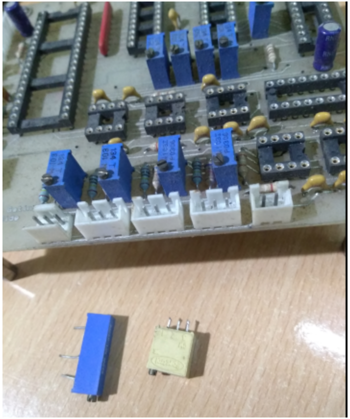


The Sallen-Key LPF: Design

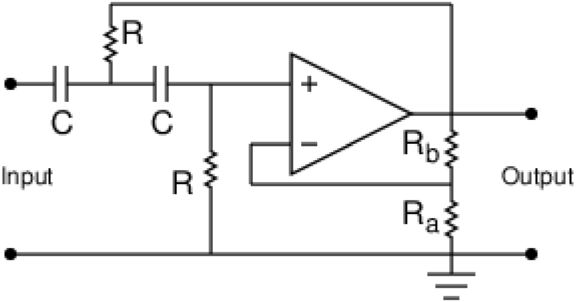
Specifications: f_0, Q, H_{desired}, and C
Compute $R = \frac{1}{2\pi f_0 C}$
Compute $K = 3 - \frac{1}{Q}$
$R_b/R_a = K - 1$. Usually one sets $R_a = 10\,\text{k}\Omega$, and then computes $R_b = (K-1)R_a$.
Compute $a = H_{\text{desired}}/K$.
Then compute $R_{\text{ser}} = R/a$ and $R_{\text{shu}} = R/(1-a)$.

Trimmer Potentiometers

Filter design requires nonstandard resistance values. They are usually implemented using *trimmer* potentiometers. The picture shows trimmers in use.



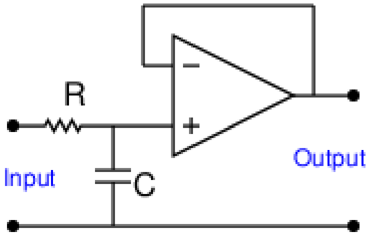
The Sallen-Key Highpass Filter



$$T(s) = \frac{Ks^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$$

$K = 1 + \frac{R_b}{R_a}$, $\omega_0 = \frac{1}{RC}$, $Q = \frac{1}{3-K}$.
Gain reduction requires a separate voltage divider followed by unity gain buffer.
Splitting of the input capacitor is not practical.

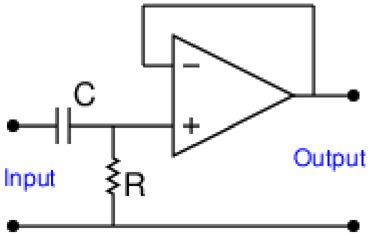
The First Order RC Lowpass Filter



$$T(s) = \frac{\omega_0}{s + \omega_0}$$

$$\omega_0 = \frac{1}{RC}$$

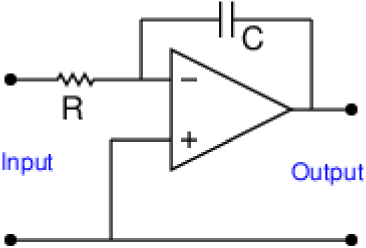
The First Order CR Highpass Filter



$$T(s) = \frac{s}{s + \omega_0}$$

$$\omega_0 = \frac{1}{RC}$$

The Inverting Integrator

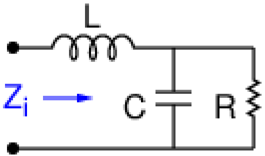


$$T(s) = -\frac{s_0}{s}$$

$$s_0 = \frac{1}{RC}$$

Remember to put a high resistance in parallel with the capacitor if the integrator is used without any other negative feedback.

Example of Impedance Matching

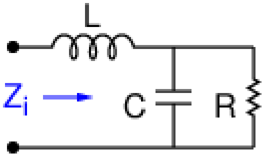


At what frequency is Z_i real?
What is its value at that frequency?

$$Z_i(s) = sL + \frac{R}{1 + sRC}$$

$$Z_i(j\omega) = j\omega L + \frac{R}{1 + j\omega RC} = j\omega L + \frac{R(1 - j\omega RC)}{1 + (\omega RC)^2}$$

Impedance Matching Analysis

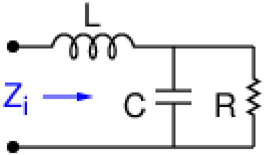


For $Z_i(j\omega)$ to be real, we require

$$\omega L = \frac{\omega R^2 C}{1 + (\omega RC)^2}$$

$$(\omega RC)^2 + 1 = \frac{R^2 C}{L}$$

Impedance Matching Results



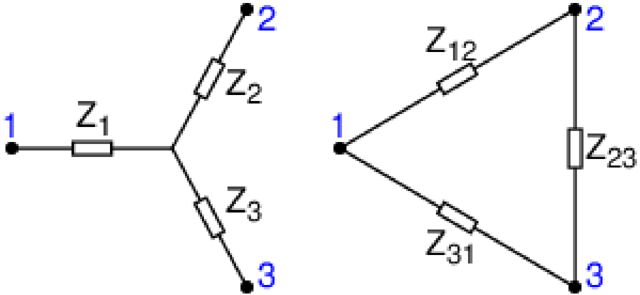
$$\omega = \frac{1}{\sqrt{LC}}\sqrt{1 - \frac{L/C}{R^2}}$$

At this ω, Z_i is

$$\frac{L/C}{R}$$

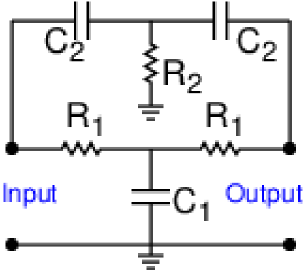
This value may be small enough for the generator to deliver more power to the load.

Star to Delta Transformation



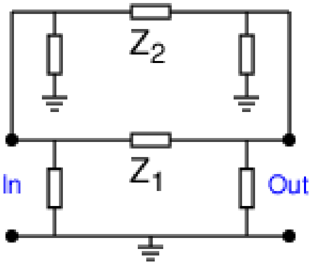
Question: When do the two networks shown look identical from the outside?
Answer: When $Z_{12} = Z_1 + Z_2 + Z_1 Z_2 / Z_3$, etc.

The Twin-T Notch Filter



At what frequency is the output zero?

Twin-T Analysis

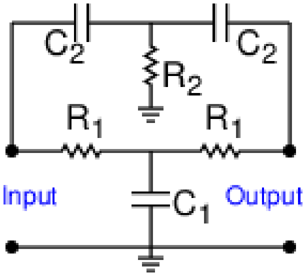


$$Z_1 = 2R_1 + sR_1^2 C_1$$

$$Z_2 = \frac{2}{sC_2} + \frac{1}{s^2 R_2 C_2^2}$$

For no transmission, we need $Z_1 = -Z_2$.

Twin-T Analysis



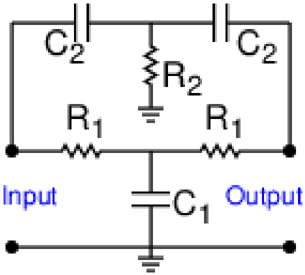
This requires

$$\omega_0^2 = \frac{2}{R_1^2 C_1 C_2}$$

and

$$\omega_0^2 = \frac{1}{2 R_1 R_2 C_2^2}$$

Twin-T Analysis



Or,

$$R_1 C_1 = 4 R_2 C_2$$

One way of achieving this is to set $R_1 = R$, $R_2 = R/2$, $C_1 = 2C$, and $C_2 = C$, so that

$$\omega_0 = \frac{1}{RC}$$

Twin-T Filter: Uses

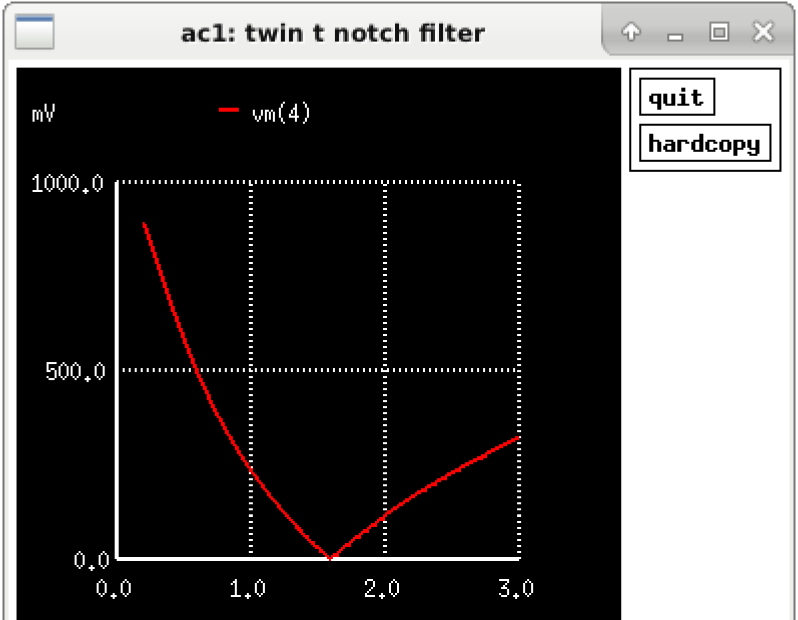
- At frequency $f_0 = 1/(2\pi RC)$, there is no transmission.
- Can be used as a notch filter.
- If used in the negative feedback path, can be part of a narrow band filter.
- This is a third order filter, not part of the mainstream.
- Much used in various forms.

Twin-T Filter: SPICE Code

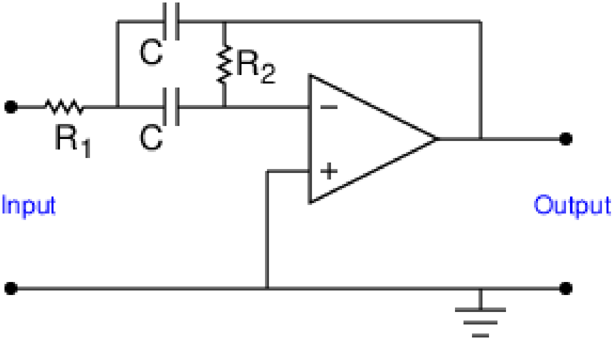
```
Twin T Notch filter
*****************************
VIN 1 0 AC 1
R1A 1 2 10k
R1B 2 4 10k
R2 3 0 5k
C1 2 0 20n
C2A 1 3 10n
C2B 3 4 10n
.AC LIN 1000 0.2k 3.0k

.CONTROL
run
plot vp(4)
plot vm(4)
.ENDCONTROL
```

Twin-T Filter: Magnitude Plot

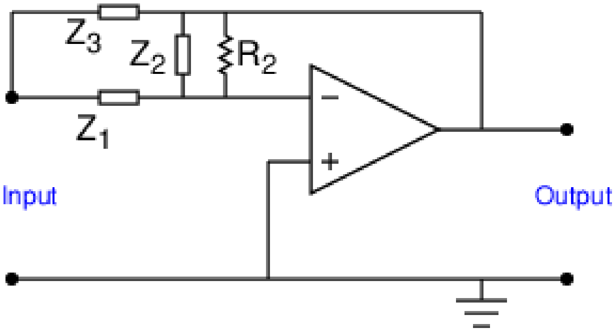


The Single Amplifier Biquad (SAB)



- This circuit acts as a bandpass filter.
- Developed by Delyiannis and Friend.

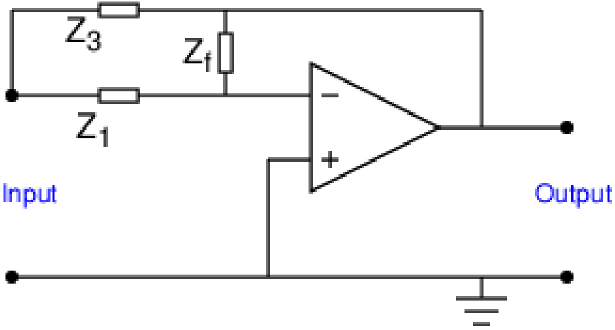
SAB Analysis



$$Z_1 = 2R_1 + \frac{1}{sC} = \frac{1 + 2sR_1C}{sC}$$

$$Z_2 = \frac{2}{sC} + \frac{1}{s^2C^2R_1} = \frac{1 + 2sR_1C}{s^2R_1C^2}$$

SAB Transfer Function



$$Z_f = Z_2||R_2 = \frac{(1+2sR_1C)R_2}{1+2sR_1C+s^2R_1R_2C^2}$$

$$T(s) = -\frac{Z_f}{Z_1} = \frac{-sR_2C}{1+2sR_1C+s^2R_1R_2C^2} = \frac{-\frac{1}{R_1C}s}{s^2+\frac{2}{R_2C}s+\frac{1}{R_1R_2C^2}}$$

SAB Parameters

We see that the SAB is a BPF.
Comparing with the standard form we get

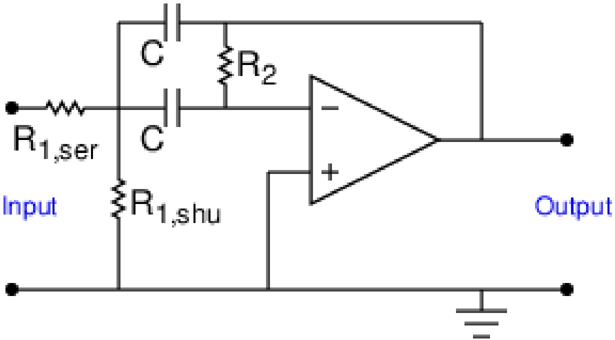
$$\omega_0 = \frac{1}{\sqrt{R_1 R_2} C}$$

$$Q = \frac{1}{2} \sqrt{\frac{R_2}{R_1}}$$

$$H = -\frac{R_2}{2R_1} = -2Q^2$$

Note that $R_2/R_1 = 4Q^2$.

SAB With Gain Reduction



SAB Design

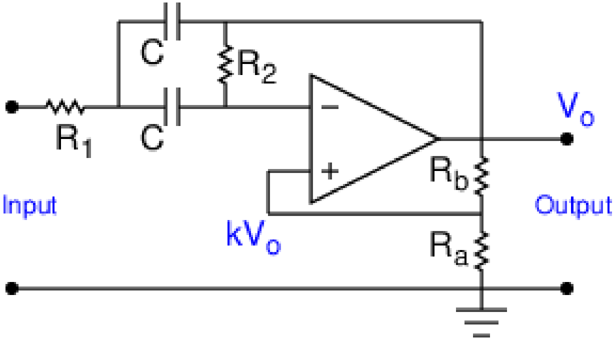
Specification: f_0, Q, H_{desired}, and C are specified.
H_{desired} should be negative.
Steps:

1. $R_2 = Q/(\pi f_0 C)$.
2. $R_1 = R_2/(4Q^2)$.
3. $a = H_{\mathrm{desired}}/(-2Q^2)$.
4. $R_{1,\mathrm{ser}} = R_1/a$.
5. $R_{1,\mathrm{shu}} = R_1/(1-a)$.

SAB Disadvantage

- $R_2/R_1 = 4Q^2$ can be quite large.
- Hard to get such high ratio inside integrated circuits.
- The remedy is to first design a low Q SAB, and then enhance its Q using positive feedback.

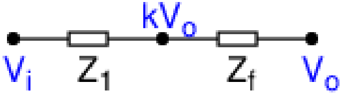
Q-Enhanced SAB



Positive feedback is used.

$$k = \frac{R_a}{R_a + R_b}$$

Q-Enhanced SAB Analysis



$$\frac{V_i - kV_o}{Z_1} = \frac{kV_o - V_o}{Z_f}$$

$$V_i - kV_o = \frac{kV_o - V_o}{Z_f/Z_1} = \frac{kV_o - V_o}{-T_0(s)} = \frac{(1-k)V_o}{T_0(s)}$$

where, $T_0(s) = -Z_f/Z_1$ is the transfer function of the original SAB.

$$V_i = kV_o + \frac{(1-k)V_o}{T_0(s)} = V_o\frac{1 - k + kT_0(s)}{T_0(s)}$$

Q-Enhanced SAB Transfer Function

So the transfer function of the Q-Enhanced SAB is

$$T(s) = \frac{V_o}{V_i} = \frac{T_0(s)}{1 - k + kT_0(s)} = \frac{\frac{1}{1-k}T_0(s)}{1 + \frac{k}{1-k}T_0(s)}$$

Let

$$\alpha = \frac{k}{1-k}$$

so that

$$k = \frac{\alpha}{1+\alpha}$$

and

$$\frac{1}{1-k} = 1 + \alpha$$

Q-Enhanced SAB Transfer Function

So we have

$$T(s) = \frac{(1+\alpha)T_0(s)}{1+\alpha T_0(s)}$$

Let us rewrite the transfer function of the original SAB, $T_0(s)$ as

$$T_0(s) = \frac{-2Q_0^2 \frac{\omega_0}{Q_0} s}{s^2 + \frac{\omega_0}{Q_0} s + \omega_0^2}$$

the original Q being rewritten as Q_0 and H being written as $-2Q_0^2$. Substitution and simplification results in

$$T(s) = \frac{-2(1+\alpha)Q_0^2 \frac{\omega_0}{Q_0} s}{s^2 + (1-2\alpha Q_0^2)\frac{\omega_0}{Q_0} s + \omega_0^2}$$

Q-Enhanced SAB Parameters

We see that for the Q-Enhanced SAB, ω_0 is unchanged,

$$Q = \frac{Q_0}{1 - 2\alpha Q_0^2}$$

and

$$H = \frac{-2Q_0^2(1+\alpha)}{1 - 2\alpha Q_0^2}$$

Selecting Q_0 and Computing α

It is recommended to use

$$Q_0 \approx 1.5$$

Compute $R_2 = Q_0/(\pi f_0 C)$, and $R_1 = R_2/(4Q_0^2)$.
Next, compute α using

$$\alpha = \frac{1 - Q_0/Q}{2Q_0^2}$$

Then compute k using

$$k = \frac{\alpha}{1 + \alpha}$$

We have $R_b/R_a = 1/k - 1$.

Gain Reduction

Compute $H = \dfrac{-2Q_0^2(1+\alpha)}{1-2\alpha Q_0^2}$.

Then compute

$$a = H_{\text{desired}}/H$$

Then compute

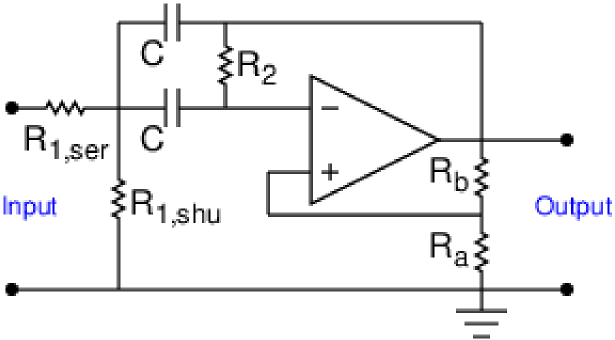
$$R_{1,\text{ser}} = R_1/a$$

and

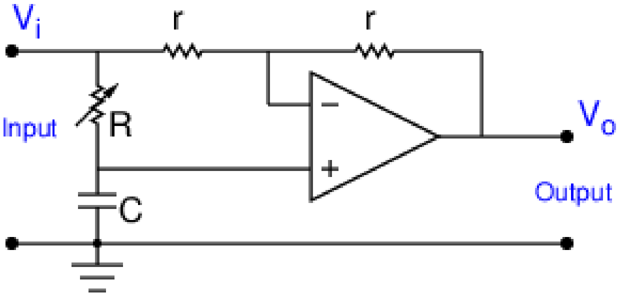
$$R_{1,\text{shu}} = R_1/(1-a)$$

The final circuit is on the next slide.

Q-Enhanced SAB with Gain Reduction

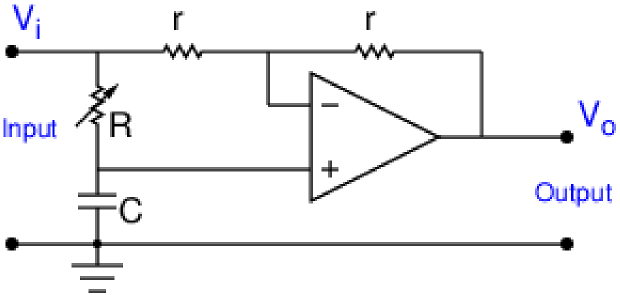


An All-Pass Filter



Used as a phase shifter.
One could make this from the difference of the transfer functions of the RC LPF and the CR HPF after suitable buffering and subtraction, but this circuit uses only one operational amplifier.

All-Pass Filter Analysis



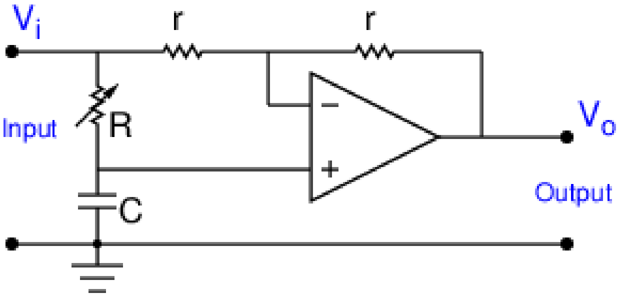
The voltage at the input pins is $(V_i + V_o)/2$.
So

$$\frac{V_i - \frac{V_i + V_o}{2}}{R} = sC\frac{V_i + V_o}{2}$$

Or,

$$\frac{V_i - V_o}{2} = sRC\frac{V_i + V_o}{2}$$

All-Pass Filter Transfer Function



Or,

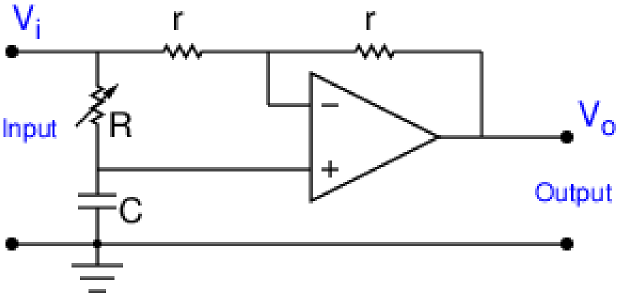
$$V_i(1 - sRC) = V_o(1 + sRC)$$

So,

$$T(s) = \frac{V_o}{V_i} = \frac{1 - sRC}{1 + sRC} = \frac{s_0 - s}{s_0 + s}$$

where $s_0 = \frac{1}{RC}$.

All-Pass Filter as a Phase Shifter



We have

$$|T(j\omega)| = 1$$

and

$$\angle T(j\omega) = -2\arctan(\omega/s_0)$$

As R changes from 0 to ∞, the phase lag changes from 0 to π.