

IN 277 Notes 4

Lossless Transformer

Gyrator

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Gyrator and Ideal Transformer

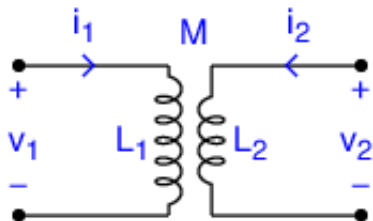
Gyrator

- Circuit related to the GIC
- Can be used to make inductor using resistor and capacitor
- Two-port network with simple v - i description
- Can be used for impedance inversion
- Non-reciprocal network

Ideal Transformer

- Two-port network with simple v - i description
- Can be used for impedance scaling
- Reciprocal network
- Follows from the concept of mutual inductance

Coupled Coils: Mutual Inductance



$$v_1 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$$

$$v_2 = M \frac{di_1}{dt} + L_2 \frac{di_2}{dt}$$

This is a model that ignores winding resistances. We shall call this configuration a *2-port lossless transformer*.

Instantaneous Power and Stored Energy

Instantaneous Power:

$$p = v_1 i_1 + v_2 i_2 = L_1 i_1 \frac{di_1}{dt} + M i_1 \frac{di_2}{dt} + M i_2 \frac{di_1}{dt} + L_2 i_2 \frac{di_2}{dt}$$

$$p = \frac{d}{dt} \left(\frac{1}{2} L_1 i_1^2 + M i_1 i_2 + \frac{1}{2} L_2 i_2^2 \right)$$

Stored Energy:

$$U = \left(\frac{1}{2} L_1 i_1^2 + M i_1 i_2 + \frac{1}{2} L_2 i_2^2 \right) = \frac{1}{2} \left(L_1 i_1^2 + 2 M i_1 i_2 + L_2 i_2^2 \right)$$

Stored Energy as a Quadratic Form

Stored Energy:

$$U = \frac{1}{2} \left(L_1 i_1^2 + 2M i_1 i_2 + L_2 i_2^2 \right) = \frac{1}{2} \begin{bmatrix} i_1 & i_2 \end{bmatrix} \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

$$U = \frac{1}{2} \mathbf{i}' \mathbf{L} \mathbf{i}$$

where $\mathbf{i} = \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$, and $\mathbf{L} = \begin{bmatrix} L_1 & M \\ M & L_2 \end{bmatrix}$.

$\mathbf{L}$ is called the *inductance matrix*.

$\mathbf{i}$ is called the *current vector*.

Stored Energy cannot be Negative

$\mathbf{L}$ needs to be positive semidefinite.

$2U = L_1 i_1^2 + 2M i_1 i_2 + L_2 i_2^2$ needs to be nonnegative no matter what the real quantities i_1 and i_2 are.

Set $i_2 = 0$ to infer that $L_1 \geq 0$.

Set $i_1 = 0$ to infer that $L_2 \geq 0$.

Assume that at least one of L_1 and L_2 is positive. If both are zero, we have no coils.

Suppose $L_1 > 0$. Then

$$2U = L_1 i_1^2 + 2M i_1 i_2 + L_2 i_2^2 = L_1 \left(i_1^2 + 2 \frac{M}{L_1} i_1 i_2 + \frac{L_2}{L_1} i_2^2 \right)$$

Condition on M

Completing squares we see that

$$2U = L_1 \left(i_1 + \frac{M}{L_1} i_2 \right)^2 + L_1 \left(\frac{L_2}{L_1} - \frac{M^2}{L_1^2} \right) i_2^2$$

Or,

$$2U = L_1 \left(i_1 + \frac{M}{L_1} i_2 \right)^2 + \left(L_2 - \frac{M^2}{L_1} \right) i_2^2$$

Setting $i_1 = -\frac{M}{L_1} i_2$, we can make the first term on the R.H.S. to be zero. Then

$2U = \left(L_2 - \frac{M^2}{L_1} \right) i_2^2$. For this to be nonnegative, we require $L_2 - \frac{M^2}{L_1} \geq 0$. Or,

$$M^2 \leq L_1 L_2$$

Range of M and the Coupling Coefficient

$$M^2 \leq L_1 L_2$$

So M must fall in a range.

$$-\sqrt{L_1 L_2} \leq M \leq \sqrt{L_1 L_2}$$

Coupling coefficient:

$$k = \frac{M}{\sqrt{L_1 L_2}}$$

This is a measure of how well the flux produced by one coil links to the other coil. From the range of M we see that

$$-1 \leq k \leq 1$$

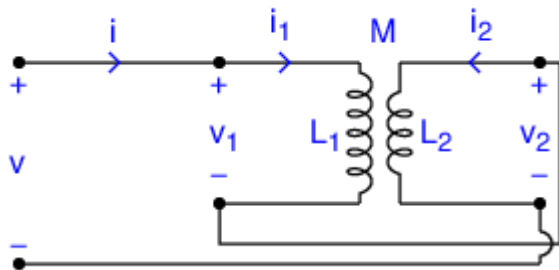
Note that in this discussion, both M and k are signed quantities. This is in contrast to what many books and softwares do. They restrict these quantities to be nonnegative only.

Inductor from Coupled Inductors

Given a 2-port lossless transformer, there are 8 ways of deriving an inductor out of it.

- Windings in series
- Windings in anti-series
- Windings in parallel
- Windings in anti-parallel
- Use primary with secondary open
- Use secondary with primary open
- Use primary with secondary shorted
- Use secondary with primary shorted

Windings in Series



Here $v = v_1 + v_2$, $i_1 = i$, and $i_2 = i$. So,

$$v = v_1 + v_2 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} + M \frac{di_1}{dt} + L_2 \frac{di_2}{dt} = L_1 \frac{di}{dt} + M \frac{di}{dt} + M \frac{di}{dt} + L_2 \frac{di}{dt}$$

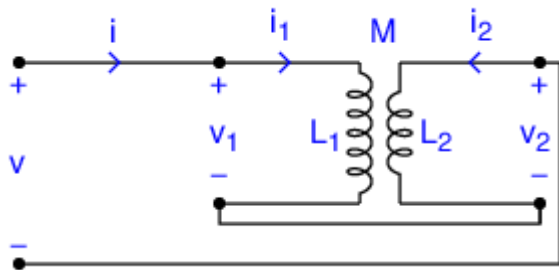
Or,

$$v = (L_1 + L_2 + 2M) \frac{di}{dt}$$

So we see that the arrangement behaves like an inductor of value

$$L_{\text{series}} = L_1 + L_2 + 2M.$$

Windings in Anti-Series



Here $v = v_1 - v_2$, $i_1 = i$, and $i_2 = -i$. So,

$$v = v_1 - v_2 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} - M \frac{di_1}{dt} - L_2 \frac{di_2}{dt}$$

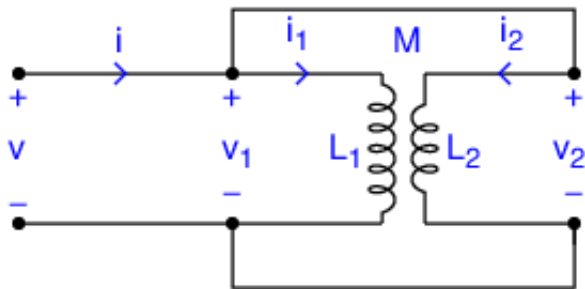
Or,

$$v = L_1 \frac{di}{dt} + M \frac{d(-i)}{dt} - M \frac{di}{dt} - L_2 \frac{d(-i)}{dt} = (L_1 + L_2 - 2M) \frac{di}{dt}$$

So we see that the arrangement behaves like an inductor of value

$$L_{\text{anti-series}} = L_1 + L_2 - 2M.$$

Windings in Parallel



Here $i = i_1 + i_2$, $v_1 = v$, and $v_2 = v$. To start with we use $v_1 = v_2$. Or,

$$L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} = M \frac{di_1}{dt} + L_2 \frac{di_2}{dt}$$

Or,

$$(L_1 - M) \frac{di_1}{dt} = (L_2 - M) \frac{di_2}{dt}$$

Windings in Parallel

So,

$$\frac{di_2}{dt} = \frac{L_1 - M}{L_2 - M} \frac{di_1}{dt}$$

Now,

$$\frac{di}{dt} = \frac{d(i_1 + i_2)}{dt} = \frac{di_1}{dt} + \frac{di_2}{dt} = \left(1 + \frac{L_1 - M}{L_2 - M}\right) \frac{di_1}{dt} = \frac{L_1 + L_2 - 2M}{L_2 - M} \frac{di_1}{dt}$$

Or,

$$\frac{di_1}{dt} = \frac{L_2 - M}{L_1 + L_2 - 2M} \frac{di}{dt}$$

So,

$$\frac{di_2}{dt} = \frac{L_1 - M}{L_2 - M} \frac{di_1}{dt} = \frac{L_1 - M}{L_1 + L_2 - 2M} \frac{di}{dt}$$

Windings in Parallel

Now,

$$v = v_1 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} = L_1 \frac{L_2 - M}{L_1 + L_2 - 2M} \frac{di}{dt} + M \frac{L_1 - M}{L_1 + L_2 - 2M} \frac{di}{dt}$$

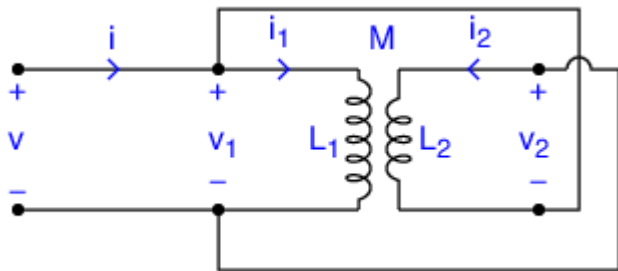
Or,

$$v = \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M} \frac{di}{dt}$$

So we see that the arrangement behaves like an inductor of value

$$L_{\text{parallel}} = \frac{L_1 L_2 - M^2}{L_1 + L_2 - 2M}$$

Windings in Anti-Parallel



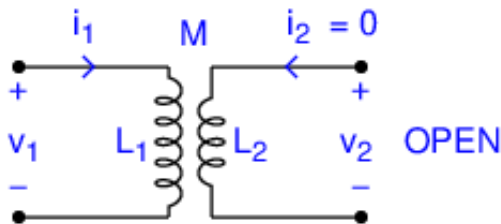
Here $i = i_1 - i_2$, $v_1 = v$, and $v_2 = -v$.

Here the arrangement behaves like an inductor of value

$$L_{\text{anti-parallel}} = \frac{L_1 L_2 - M^2}{L_1 + L_2 + 2M}$$

the derivation being similar to the parallel case.

One Winding Open

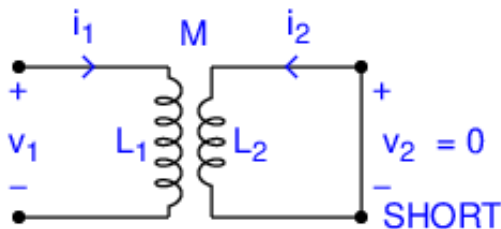


When the secondary is open, $i_2 = 0$. So $v_1 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} = L_1 \frac{di_1}{dt}$.

So the primary looks like an inductor of value L_1 , which is what we expect.

Similar analysis shows that when the primary is open, the secondary looks like an inductor of value L_2 .

One Winding Shorted



When the secondary is shorted,

$$v_2 = M \frac{di_1}{dt} + L_2 \frac{di_2}{dt} = 0$$

So

$$\frac{di_2}{dt} = -\frac{M}{L_2} \frac{di_1}{dt}$$

One Winding Shorted

Then

$$v_1 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} = L_1 \frac{di_1}{dt} - M \frac{M}{L_2} \frac{di_1}{dt} = \frac{L_1 L_2 - M^2}{L_2} \frac{di_1}{dt}$$

So with the secondary shorted, the primary behaves like an inductor of value

$$L_{1,\text{leakage}} = \frac{L_1 L_2 - M^2}{L_2} = L_1 - \frac{M^2}{L_2}$$

Similar analysis shows that with the primary shorted, the secondary behaves like an inductor of value

$$L_{2,\text{leakage}} = \frac{L_1 L_2 - M^2}{L_1} = L_2 - \frac{M^2}{L_1}$$

Condition for a Fixed Voltage Ratio

When is v_1/v_2 constant regardless of what the transformer currents are?

This is same as asking the following question.

Given constants α , β , γ , and δ , when is $\frac{\alpha x + \beta y}{\gamma x + \delta y}$ independent of x and y ?

The answer is that the ratio will be independent of x and y , when

$$\frac{\alpha}{\gamma} = \frac{\beta}{\delta}.$$

Note the correspondences: $\alpha = L_1$, $\beta = M$, $\gamma = M$, $\delta = L_2$, $x = \frac{di_1}{dt}$, and $y = \frac{di_2}{dt}$.

So for a fixed voltage ratio v_1/v_2 , we require,

$$\frac{L_1}{M} = \frac{M}{L_2}$$

Perfect Transformer

When we have

$$\frac{L_1}{M} = \frac{M}{L_2}$$

or, what is the same thing,

$$M^2 = L_1 L_2$$

or, what is the same thing,

$$k = \pm 1$$

the ratio v_1/v_2 is independent of the load, and such a transformer is called a *perfect transformer*. In a perfect transformer, the primary and secondary are perfectly coupled. The leakage inductances are zero in this case.

Voltage Ratio of a Perfect Transformer

$$\frac{v_1}{v_2} = \frac{L_1}{M} = \frac{M}{L_2} = \pm \sqrt{\frac{L_1}{L_2}} = n$$

The sign is to be taken as the sign of M .

The number n is known as the *turns ratio*.

A perfect transformer is completely characterized by two parameters:

- ① the voltage ratio n , and
- ② any one of the inductances

This is a reduction from the three parameters needed to specify an arbitrary 2-port lossless transformer.

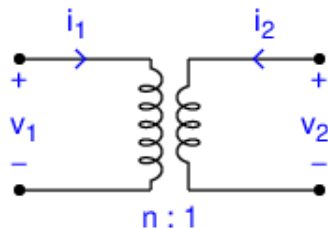
Ideal Transformer

The perfect transformer looks like a load of inductance L_1 on the primary side, even if there is no load on the secondary. So it takes reactive current even on no load.

Now consider a situation in which we let $L_1 \rightarrow \infty$ keeping $n = L_1/M = M/L_2$ fixed. Then we have a perfect transformer which does not load the primary if there is no load on the secondary.

Such a transformer is called an *ideal transformer*. It is characterized by just one parameter.

That is the turns ratio $n = v_1/v_2 = \frac{L_1}{M} = \frac{M}{L_2}$.



Fixed Current Ratio

$$v_1 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$$

Or,

$$\frac{v_1}{L_1} = \frac{di_1}{dt} + \frac{M}{L_1} \frac{di_2}{dt}$$

If we let $L_1 \rightarrow \infty$ keeping $n = L_1/M = M/L_2$ fixed, the L.H.S. will tend to 0, and we will have

$$\frac{di_1}{dt} = -\frac{M}{L_1} \frac{di_2}{dt} = -\frac{1}{n} \frac{di_2}{dt}$$

If we disregard the DC parts of the currents, we have

$$i_1 = -\frac{1}{n} i_2$$

for the ideal transformer, in addition to the $v_1 = nv_2$ given by the fact that an ideal transformer is also a perfect transformer.

Ideal Transformer: ABCD Matrix

Noting that $V_{\text{in}} = V_1$, $V_{\text{out}} = V_2$, $I_{\text{in}} = I_1$, and $I_{\text{out}} = -I_2$, we can write for the ideal transformer

$$V_{\text{in}} = nV_{\text{out}}$$

and

$$I_{\text{in}} = \frac{1}{n}I_{\text{out}}$$

So its ABCD matrix is

$$\begin{bmatrix} n & 0 \\ 0 & \frac{1}{n} \end{bmatrix}$$

Note that with the ideal transformer,

$$Z_{\text{in}} = \frac{AZ_{\text{load}} + B}{CZ_{\text{load}} + D} = \frac{nZ_{\text{load}} + 0}{0Z_{\text{load}} + \frac{1}{n}} = n^2Z_{\text{load}}$$

Ideal Transformer: Points to Note

- No derivatives needed to describe it.
- It is a non-reactive element.
- Characterized by just one parameter n .
- Can be used for impedance transformation.
- Other transformers can be modelled in terms of the ideal transformer.

One major use of a transformer is to provide isolation.

Transferring Series Element

Note that

$$\begin{bmatrix} n & 0 \\ 0 & \frac{1}{n} \end{bmatrix} \begin{bmatrix} 1 & Z \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} n & nZ \\ 0 & \frac{1}{n} \end{bmatrix} = \begin{bmatrix} 1 & n^2 Z \\ 0 & 1 \end{bmatrix} \begin{bmatrix} n & 0 \\ 0 & \frac{1}{n} \end{bmatrix}$$

So we have the following equivalence.



Transferring Shunt Element

Note that

$$\begin{bmatrix} n & 0 \\ 0 & \frac{1}{n} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ Y & 1 \end{bmatrix} = \begin{bmatrix} n & 0 \\ Y/n & \frac{1}{n} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ Y/n^2 & 1 \end{bmatrix} \begin{bmatrix} n & 0 \\ 0 & \frac{1}{n} \end{bmatrix}$$

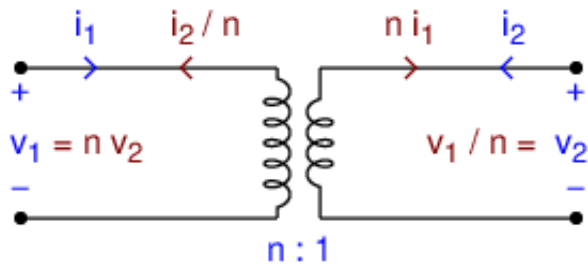
So we have the following equivalence.



Summary: Transferring Element

- Transferring an impedance from secondary to primary multiplies it by n^2 .
- Transferring an admittance from secondary to primary divides it by n^2 .
- This can be used repeatedly.

One More Look at the Ideal Transformer



Model for a Lossless Transformer

We start with

$$v_1 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$$

$$v_2 = M \frac{di_1}{dt} + L_2 \frac{di_2}{dt}$$

Let $n = \frac{M}{L_2}$, so that $nv_2 = \frac{M^2}{L_2} \frac{di_1}{dt} + M \frac{di_2}{dt}$.

Then

$$v_1 = \left(L_1 - \frac{M^2}{L_2} \right) \frac{di_1}{dt} + nv_2$$

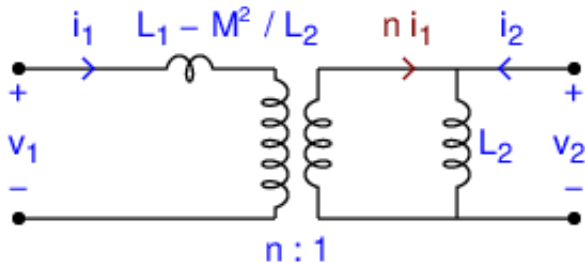
This suggests a primary side series inductor of value $L_1 - \frac{M^2}{L_2}$ followed by an ideal transformer with ratio $n = \frac{M}{L_2}$. But there is one more inductor.

Model for a Lossless Transformer

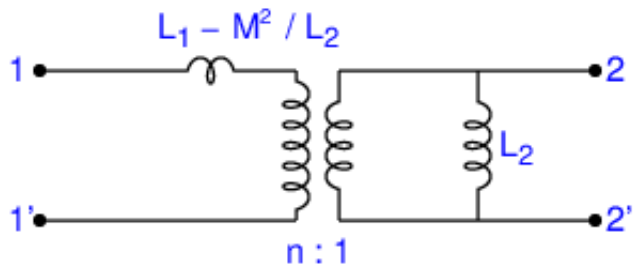
Note that

$$v_2 = M \frac{di_1}{dt} + L_2 \frac{di_2}{dt} = L_2 \frac{d \left(i_2 + \frac{M}{L_2} i_1 \right)}{dt} = L_2 \frac{d (i_2 + n i_1)}{dt}$$

This suggests that there is a shunt inductor on the secondary side with value L_2 .

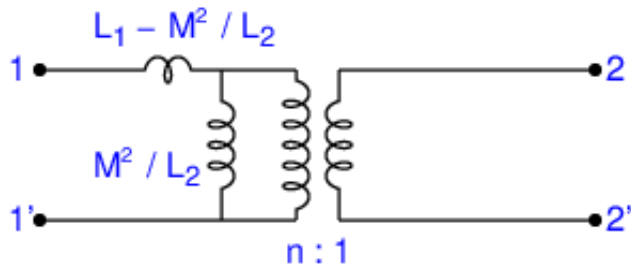


Lossless Transformer Model 1



$$n = \frac{M}{L_2}$$

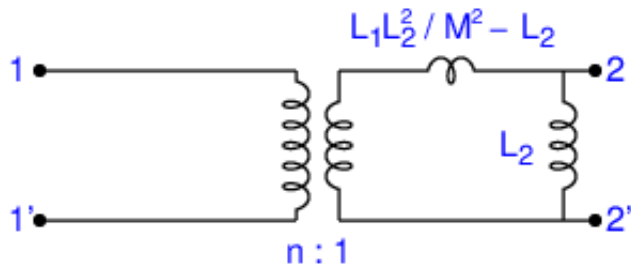
Lossless Transformer Model 2



$$n = \frac{M}{L_2}$$

This model was obtained from Model 1 by transferring L_2 to the primary side.

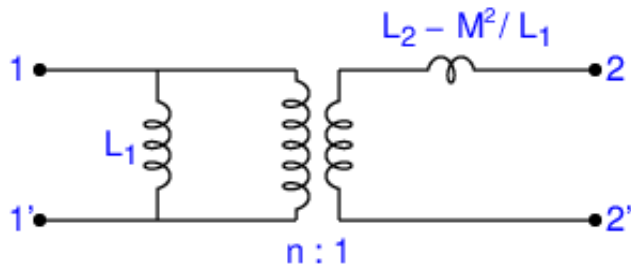
Lossless Transformer Model 3



$$n = \frac{M}{L_2}$$

This model was obtained from Model 1 by transferring $L_1 - M^2/L_2$ to the secondary side.

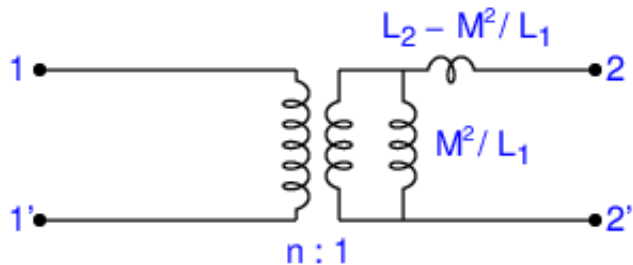
Lossless Transformer Model 4



$$n = \frac{L_1}{M}$$

This model was obtained like Model 1, but starting from the secondary side.

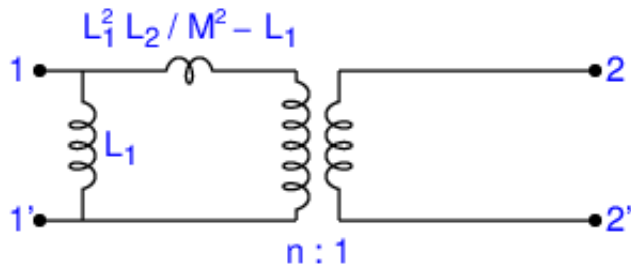
Lossless Transformer Model 5



$$n = \frac{L_1}{M}$$

This model was obtained from Model 4 by transferring L_1 to the secondary side.

Lossless Transformer Model 6



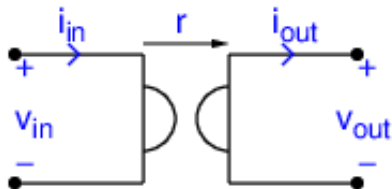
$$n = \frac{L_1}{M}$$

This model was obtained from Model 4 by transferring $L_2 - M^2/L_1$ to the primary side.

Bernard D. H. Tellegen (1900-1990)

- Dutch Electrical Engineer
- Philips
- Pentode (1926)
- Gyrator (1948)
- Tellegen's Theorem (1952)

The Gyrator

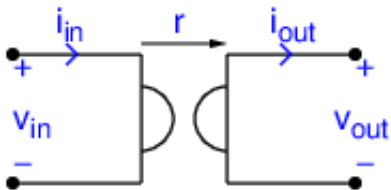


$$v_{\text{in}} = r i_{\text{out}}$$

$$i_{\text{in}} = v_{\text{out}} / r$$

It is defined by a single parameter r , which is called its *gyration resistance*.
As $v_{\text{in}} i_{\text{in}} = r i_{\text{out}} v_{\text{out}} / r = v_{\text{out}} i_{\text{out}}$, it is a lossless device.

The Gyrator: ABCD Matrix



$$V_{\text{in}} = r i_{\text{out}}$$

$$i_{\text{in}} = V_{\text{out}}/r$$

So its ABCD matrix is

$$\begin{bmatrix} 0 & r \\ 1/r & 0 \end{bmatrix}$$

The Gyrator: Impedance Inversion

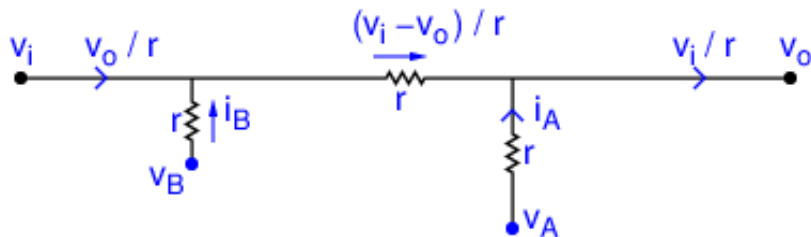
For a load Z_L connected at the output port of the gyrator, the input impedance is

$$Z_I = \frac{AZ_L + B}{CZ_L + D} = \frac{0Z_L + r}{(1/r)Z_L + 0} = \frac{r^2}{Z_L}$$

So the input impedance is a real quantity divided by the load impedance.

Connecting a capacitor of value C at the load will make the input behave like an inductor of value $L = r^2 C$.

Making a Gyrator



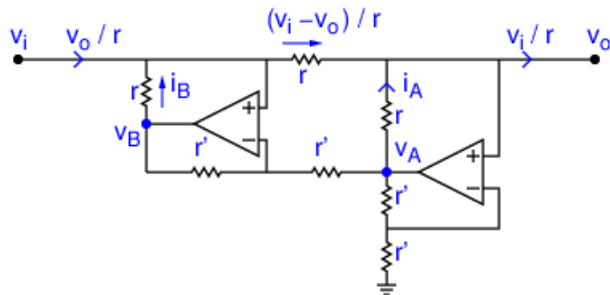
$$i_A = v_i/r - (v_i - v_o)/r = v_o/r$$

$$v_A = v_o + ri_A = v_o + rv_o/r = 2v_o$$

$$i_B = (v_i - v_o)/r - v_o/r = (v_i - 2v_o)/r$$

$$v_B = v_i + ri_B = v_i + r(v_i - 2v_o)/r = 2(v_i - v_o)$$

The Gyrator: One Realization



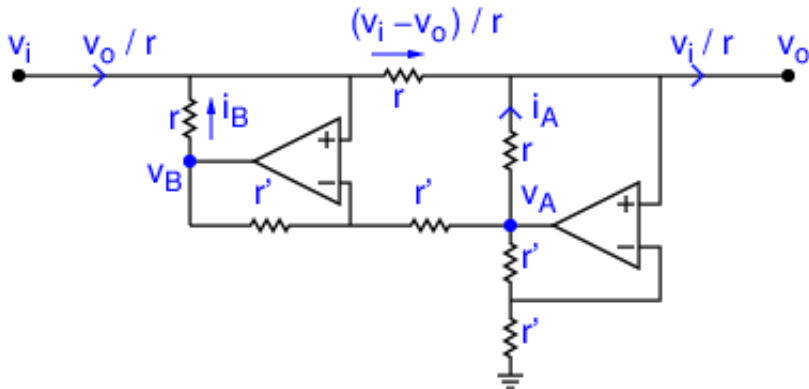
$$i_A = v_i/r - (v_i - v_o)/r = v_o/r$$

$$v_A = v_o + ri_A = v_o + rv_o/r = 2v_o$$

$$i_B = (v_i - v_o)/r - v_o/r = (v_i - 2v_o)/r$$

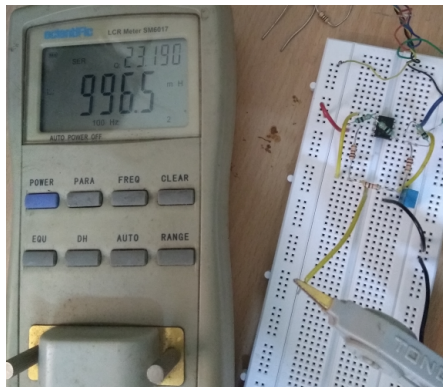
$$v_B = v_i + ri_B = v_i + r(v_i - 2v_o)/r = 2(v_i - v_o)$$

The Gyrator: Components



- Two operational amplifiers
- 7 resistors
- Stable even if components are not perfectly matched.

The Gyration: Demonstration



Tried with $r = 1.0 \text{ k}\Omega$, $r' = 10.0 \text{ k}\Omega$, $C = 1.0 \text{ }\mu\text{F}$.

Operational amplifier: NE 5532

L should have been $1.0 \text{ H} = 1000.0 \text{ mH}$.

Got $L = 996.5 \text{ mH}$, $Q = 23.19$ at 100 Hz .

Ideal Transformer: Two Gytrators in Cascade

$$\begin{bmatrix} 0 & r_1 \\ 1/r_1 & 0 \end{bmatrix} \begin{bmatrix} 0 & r_2 \\ 1/r_2 & 0 \end{bmatrix} = \begin{bmatrix} r_1/r_2 & 0 \\ 0 & r_2/r_1 \end{bmatrix} = \begin{bmatrix} n & 0 \\ 0 & 1/n \end{bmatrix}$$

where, $n = \frac{r_1}{r_2}$.

So the gyrator eliminates not only the inductor, but also the mutual inductor. Of course, this is in theory only. It cannot provide the galvanic isolation that a transformer does.

Floating Inductor

$$\begin{bmatrix} 0 & r \\ 1/r & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ sC & 1 \end{bmatrix} \begin{bmatrix} 0 & r \\ 1/r & 0 \end{bmatrix} = \begin{bmatrix} srC & r \\ 1/r & 0 \end{bmatrix} \begin{bmatrix} 0 & r \\ 1/r & 0 \end{bmatrix} = \begin{bmatrix} 1 & sr^2C \\ 0 & 1 \end{bmatrix}$$

Gyrator followed by Shunt Capacitor followed by Gyrator

= Series Inductor

Equivalent Inductance: $L = r^2 C$

Numerical Example: If $r = 1 \text{ k}\Omega$, and $C = 0.1 \text{ }\mu\text{F}$, then $L = 0.1 \text{ H}$.

Increasing r to $10 \text{ k}\Omega$ will make $L = 10.0 \text{ H}$.

The Gyrator: Properties

- Lossless
- Non-reciprocal
- Non-reactive
- Converts capacitor to inductor.
- Converts a voltage source to a current source.
- Inverts the v-i characteristic of a non-linear device.
- Provides new two-port networks: isolator and circulator.

The Active Gyrator: Limitations

- Cannot handle large voltages or currents.
- Cannot provide galvanic isolation.

The Active Gyrator: Advantages

- Can be very compact.
- Has made telephone circuits smaller.
- Has greatly improved low frequency analogue filters.