IN 277 Notes 5
T Model of the Lossless Transformer
Pole Locations
Butterworth Lowpass Filter

A. Mohanty

Department of Instrumentation and Applied Physics
Indian Institute of Science
Bangalore 560012

October 23, 2025

Z Matrix of a Lossless Transformer

We start with

$$v_1 = L_1 \frac{\mathrm{d}i_1}{\mathrm{d}t} + M \frac{\mathrm{d}i_2}{\mathrm{d}t}$$

$$v_2 = M \frac{\mathrm{d}i_1}{\mathrm{d}t} + L_2 \frac{\mathrm{d}i_2}{\mathrm{d}t}$$

Assuming e^{st} time dependence converts differentiation to multiplication by s.

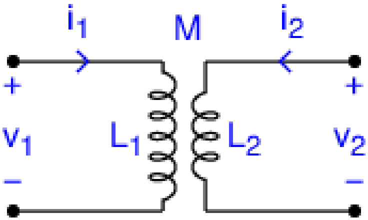
$$V_1 = sL_1 I_1 + sMI_2$$

$$V_2 = sMI_1 + sL_2 I_2$$

So the Z matrix of a lossless transformer is

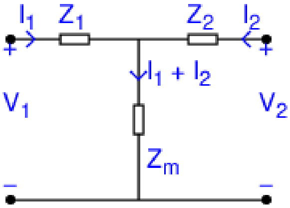
$$Z = \begin{bmatrix} sL_1 & sM \\ sM & sL_2 \end{bmatrix}$$

Z Matrix of a Lossless Transformer



$$Z = \begin{bmatrix} sL_1 & sM \\ sM & sL_2 \end{bmatrix}$$

Z Matrix of a T Network

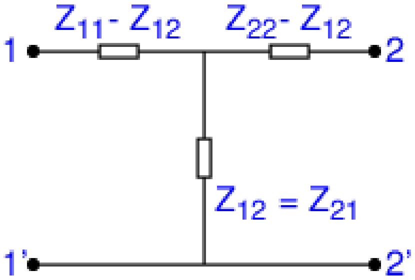


$$V_1 = I_1 Z_1 + (I_1 + I_2) Z_m = (Z_1 + Z_m) I_1 + Z_m I_2$$

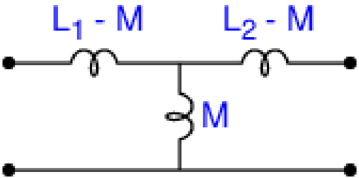
$$V_2 = I_2 Z_2 + (I_1 + I_2) Z_m = Z_m I_1 + (Z_2 + Z_m) I_2$$

$$Z = \begin{bmatrix} Z_1 + Z_m & Z_m \\ Z_m & Z_2 + Z_m \end{bmatrix}$$

T Network for a Symmetric Z Matrix

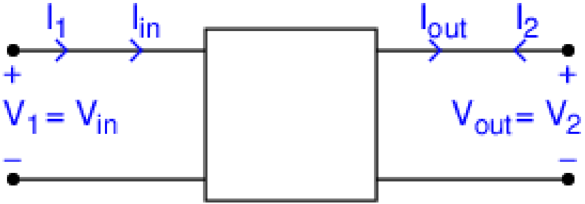


T Network for a Lossless Transformer



- Applicable only when both sides share a common terminal.
- Does not show any isolation unlike the models presented earlier.
- Still useful for many calculations.

Z Matrix from ABCD Matrix



$$V_{\text{in}} = \mathcal{A}V_{\text{out}} + \mathcal{B}I_{\text{out}}$$

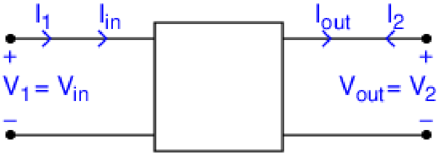
$$I_{\text{in}} = \mathcal{C}V_{\text{out}} + \mathcal{D}I_{\text{out}}$$

Note that $V_1 = V_{\text{in}}$, $V_2 = V_{\text{out}}$, $I_1 = I_{\text{in}}$, and $I_2 = -I_{\text{out}}$.

$$V_1 = \mathcal{A}V_2 - \mathcal{B}I_2$$

$$I_1 = \mathcal{C}V_2 - \mathcal{D}I_2$$

Z Matrix from ABCD Matrix



From the last equation

$$V_2 = \frac{1}{\mathcal{C}} I_1 + \frac{\mathcal{D}}{\mathcal{C}} I_2$$

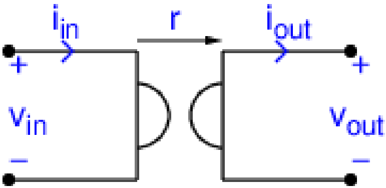
Using this in the V_1 equation, we get

$$V_1 = \frac{\mathcal{A}}{\mathcal{C}} I_1 + \frac{\mathcal{A}\mathcal{D}}{\mathcal{C}} I_2 - \mathcal{B} I_2 = \frac{\mathcal{A}}{\mathcal{C}} I_1 + \frac{\mathcal{A}\mathcal{D} - \mathcal{B}\mathcal{C}}{\mathcal{C}} I_2$$

Comparing with the definition of the Z matrix we get

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} \mathcal{A}/\mathcal{C} & (\mathcal{A}\mathcal{D} - \mathcal{B}\mathcal{C})/\mathcal{C} \\ 1/\mathcal{C} & \mathcal{D}/\mathcal{C} \end{bmatrix}$$

Z Matrix of a Gyrator



The ABCD matrix of a gyrator is

$$\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} = \begin{bmatrix} 0 & r \\ 1/r & 0 \end{bmatrix}$$

So its Z matrix is

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} 0 & -r \\ r & 0 \end{bmatrix}$$

It is not symmetric. The gyrator is not a reciprocal network.

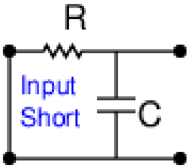
Pole of Transfer Function

- $T(s)$ is ratio of output to input for e^{st} time dependence of all voltages and currents.
- For lumped element networks, $T(s)$ is the ratio of two real polynomials in s.
- So $T(s) = \dfrac{A(s)}{B(s)}$, where $A(s)$ and $B(s)$ are real polynomials in s.
- For some values of s, the denominator polynomial $B(s)$ can be 0. Such values of s are called the *poles* of $T(s)$.
- As we approach a pole, $|T(s)|$ tends to infinity.
- One way of understanding a pole: We get non-zero output even if the input is zero.

Pole Location and Stability

- If $s_1, s_2, \ldots, s_n$ are the poles of a transfer function $T(s)$, a natural response of the form $a_1 e^{s_1 t} + a_2 e^{s_2 t} + \ldots a_n e^{s_n t}$, may be observed at the output, even if the input is zero. Here $a_1, a_2, \ldots, a_n$ are arbitrary constants.
- If the transfer function of a system has a pole in the right half plane, then the system is unstable.
- For a stable system, all poles of the transfer function should be in the left half of the s plane (LHP), or on the imaginary axis.

Pole of Transfer Function: Example



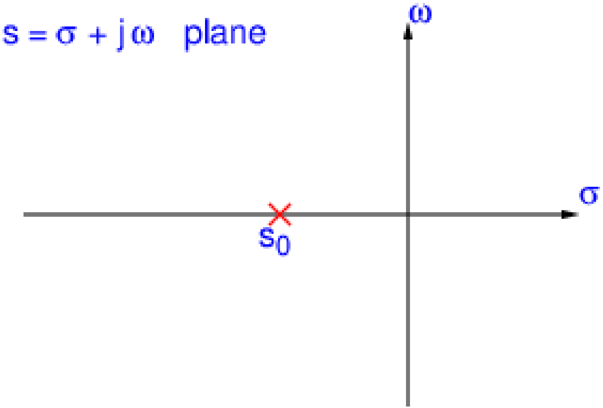
$$T(s) = \frac{\omega_0}{s + \omega_0}$$

where $\omega_0 = \dfrac{1}{RC}$.

Pole at $s_0 = -\omega_0 = -\dfrac{1}{RC}$.

Sure enough we will see $e^{-\frac{1}{RC}t} = e^{s_0 t}$ time dependence at the output, if there is charge on the capacitor.

Pole Location of First-Order Filters



Pole for both the LPF $T(s) = \dfrac{\omega_0}{s + \omega_0}$, and the HPF $T(s) = \dfrac{s}{s + \omega_0}$.

Pole at $s_0 = -\omega_0$.

Pole Location of Second-Order Filters

$$T_{\mathrm{LPF}}(s) = \frac{H\omega_0^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$$

$$T_{\mathrm{BPF}}(s) = \frac{H\frac{\omega_0}{Q}s}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$$

$$T_{\mathrm{HPF}}(s) = \frac{Hs^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$$

In each case, the denominator is

$$s^2 + \frac{\omega_0}{Q}s + \omega_0^2$$

Pole Location of Second-Order Filters

The denominator is

$$s^2 + \frac{\omega_0}{Q}s + \omega_0^2$$

The zeros of this polynomial are the poles.

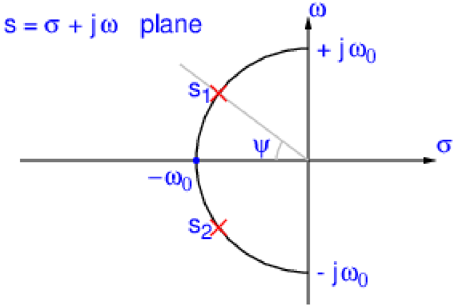
$$s_1 = -\frac{\omega_0}{2Q} + \omega_0\sqrt{\frac{1}{4Q^2} - 1}$$

and

$$s_2 = -\frac{\omega_0}{2Q} - \omega_0\sqrt{\frac{1}{4Q^2} - 1}$$

The product of the poles is $s_1 s_2 = \omega_0^2$.
The location of the poles and the nature of the circuit depends on the value of Q.

The Underdamped Case: $Q > 1/2$



If $Q > 1/2$,

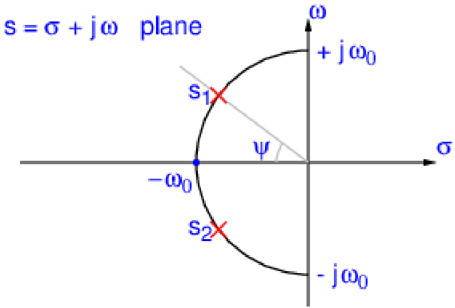
$$s_1 = -\frac{\omega_0}{2Q} + j\omega_0\sqrt{1 - \frac{1}{4Q^2}}$$

and

$$s_2 = -\frac{\omega_0}{2Q} - j\omega_0\sqrt{1 - \frac{1}{4Q^2}}$$

Then we have a complex conjugate pair of poles.

The Underdamped Case: $Q > 1/2$

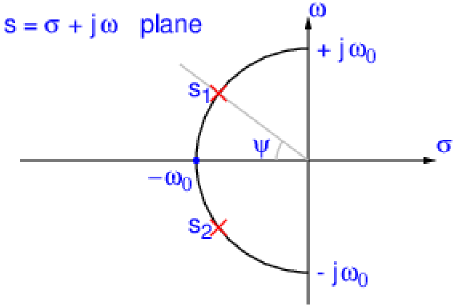


The real part of both s_1 and s_2 is $\sigma_1 = -\dfrac{\omega_0}{2Q}$, which is negative.

The imaginary part of s_1 is $\omega_1 = \omega_0\sqrt{1 - \frac{1}{4Q^2}}$.

Since $\sigma_1^2 + \omega_1^2 = \omega_0^2$, the poles lie on a circle of radius ω_0.

The Underdamped Case: $Q > 1/2$



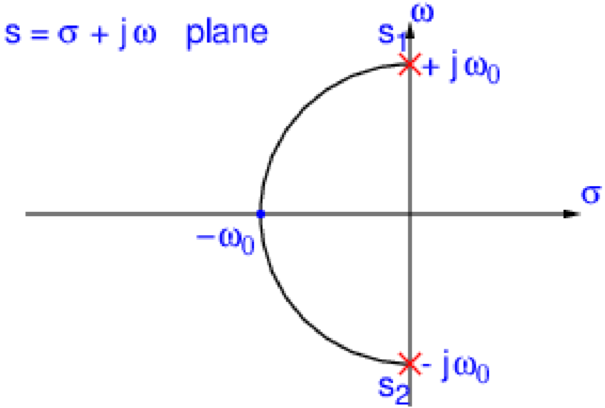
The angle ψ measured from the negative real axis to the upper pole s_1 is given by

$$\cos\psi = \frac{-\sigma_1}{\omega_0} = \frac{1}{2Q}$$

So

$$Q = \frac{1}{2\cos\psi}$$

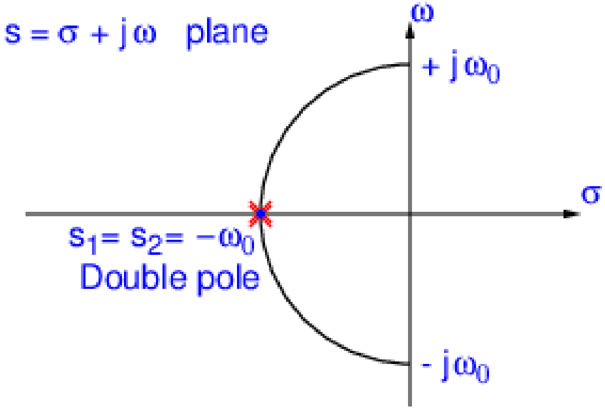
The Undamped Case: $Q \to \infty$



$$s_1 = +j\omega_0$$

$$s_2 = -j\omega_0$$

The Critically Damped Case: $Q = 1/2$

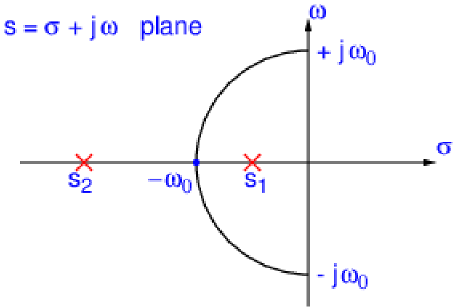


$$s_1 = -\omega_0$$

$$s_2 = -\omega_0$$

We have equal real negative roots. So we have a *double* pole.

The Overdamped Case: $Q < 1/2$

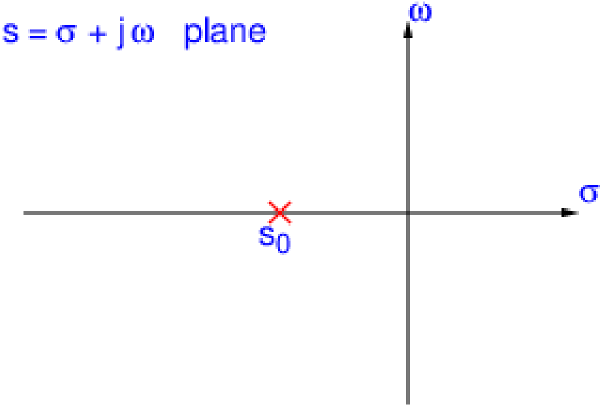


$$s_1 = -\frac{\omega_0}{2Q} + \omega_0\sqrt{\frac{1}{4Q^2} - 1}$$

$$s_2 = -\frac{\omega_0}{2Q} - \omega_0\sqrt{\frac{1}{4Q^2} - 1}$$

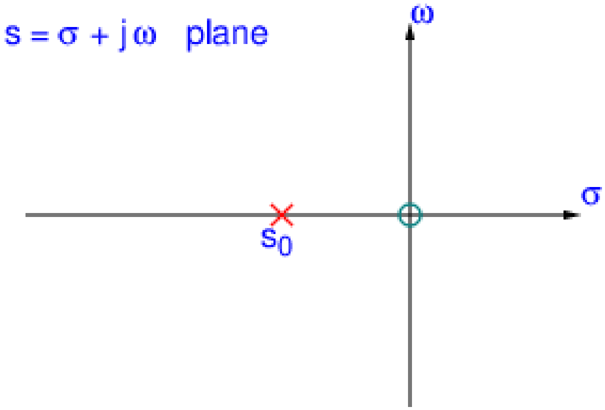
Both roots are real, negative, and unequal. We still have $s_1 s_2 = \omega_0^2$.

Pole-Zero Diagram: First-order LPF



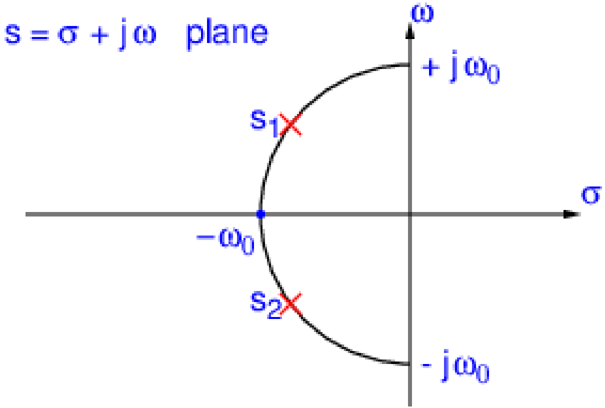
$$T(s) = \frac{\omega_0}{s + \omega_0}$$

Pole-Zero Diagram: First-order HPF



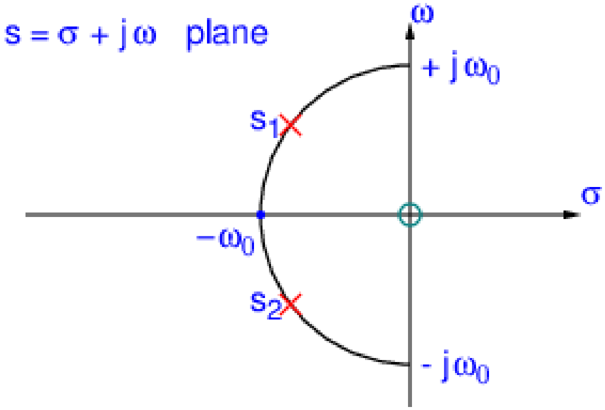
$$T(s) = \frac{s}{s + \omega_0}$$

Pole-Zero Diagram: Second-order LPF



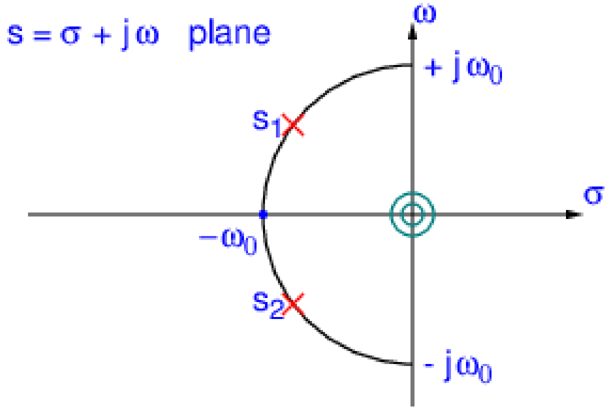
$$T(s) = \frac{H\omega_0^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$$

Pole-Zero Diagram: Second-order BPF



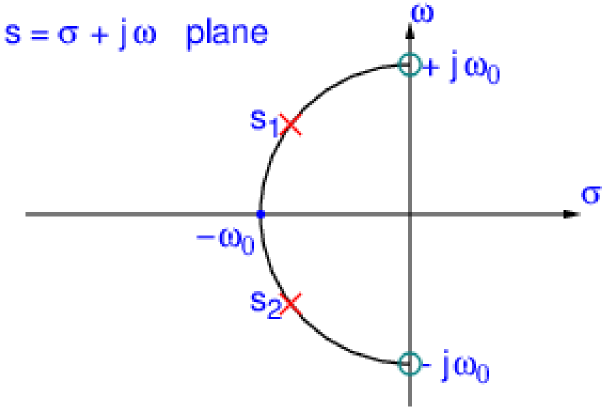
$$T(s) = \frac{H\frac{\omega_0}{Q}s}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$$

Pole-Zero Diagram: Second-order HPF



$$T(s) = \frac{Hs^2}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$$

Pole-Zero Diagram: Second-order Notch Filter



$$T(s) = \frac{H(s^2 + \omega_0^2)}{s^2 + \frac{\omega_0}{Q}s + \omega_0^2}$$

Active Filter Design Plan

- Learn Lowpass Filter Design first.
- Learn Frequency Transformation Theory.
- Apply frequency transformation to design other types of filters from existing LPF designs.

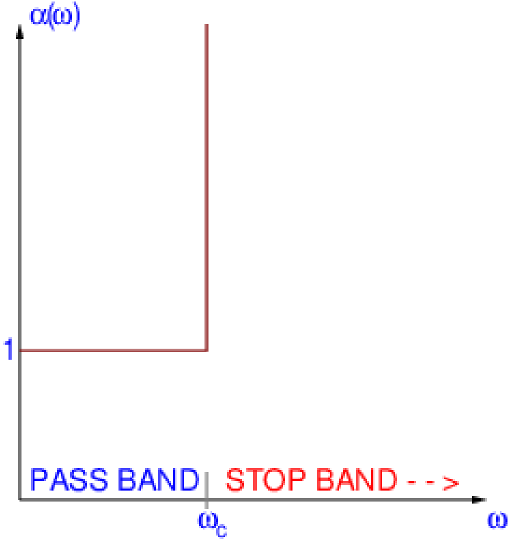
Filter Design Terminology: Loss Function $\alpha(\omega)$

Power Gain Function: Output power divided by input power
(Power) Loss Function: Input power divided by output power

$$\alpha(\omega) = \frac{1}{|T(j\omega)|^2}$$

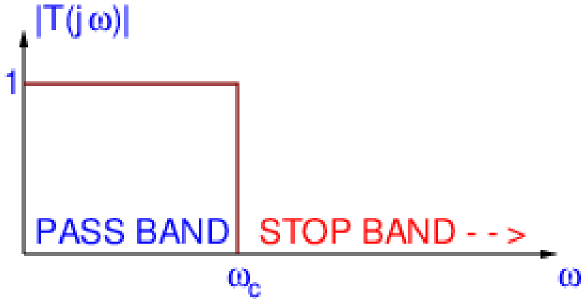
Pass band: Loss function $\alpha(\omega)$ is close to 1.
Stop band: Loss function $\alpha(\omega)$ is large.

Filter Design Terminology: Brick Wall LPF: $\alpha(\omega)$



Not realizable due to continuous nature of $T(s)$.

Filter Design Terminology: Brick Wall LPF: $|T(j\omega)|$



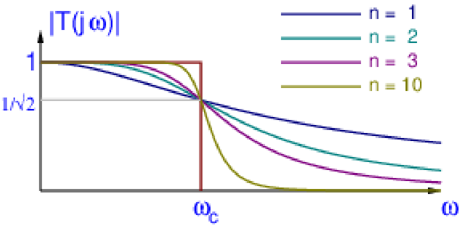
Filter Design: $|T(j\omega)|^2$

- For lumped circuit networks, $T(s)$ is a ratio of two polynomials with real coefficients.
- So $\overline{T(s)} = T(\overline{s})$.
- So $|T(j\omega)|^2 = \overline{T(j\omega)}T(j\omega) = T(\overline{j\omega})T(j\omega) = T(-j\omega)T(j\omega) = T(-s)T(s)|_{s=j\omega}$.

Butterworth Filter

- Proposed first in 1930 by the British physicist Stephen Butterworth (1885 - 1958).
- Butterworth LPF: $\alpha(\omega) = 1 + \left(\frac{\omega}{\omega_c}\right)^{2n}$
- Defined by two parameters: Order n and cutoff frequency f_c. Note that $\omega_c = 2\pi f_c$.

Butterworth Lowpass Filter $|T(j\omega)|$



$$\alpha(\omega) = 1 + \left(\frac{\omega}{\omega_c}\right)^{2n}$$

$$|T(j\omega)|^2 = \frac{1}{\alpha(\omega)} = \frac{1}{1 + \left(\frac{\omega}{\omega_c}\right)^{2n}}$$

Defined by two parameters: $\omega_c = 2\pi f_c$, and order n.
$|T(j\omega)| = 1$ at $\omega = 0$, $|T(j\omega)| = 1/\sqrt{2}$ at $\omega = \omega_c$.

Butterworth LPF: Poles of $T(s)T(-s)$

$$T(s)T(-s) = \frac{1}{1 + \left(\frac{s}{j\omega_c}\right)^{2n}}$$

Poles of $T(s)T(-s)$ are given by

$$1 + \left(\frac{s}{j\omega_c}\right)^{2n} = 0$$

Or

$$\left(\frac{s}{j\omega_c}\right)^{2n} = -1 = e^{j(2k+1)\pi}$$

where k is an integer.

Butterworth LPF: Poles of $T(s)T(-s)$

So the pole with index k is given by

$$\frac{s_k}{j\omega_c} = e^{j\frac{(2k+1)\pi}{2n}} = e^{j\frac{(k+1/2)\pi}{n}}$$

Or,

$$s_k = j\omega_c e^{j\frac{(k+1/2)\pi}{n}} = \omega_c e^{j\left(\frac{\pi}{2}+\frac{(k+1/2)\pi}{n}\right)}$$

Or,

$$s_k = \omega_c e^{j\phi_k}$$

where

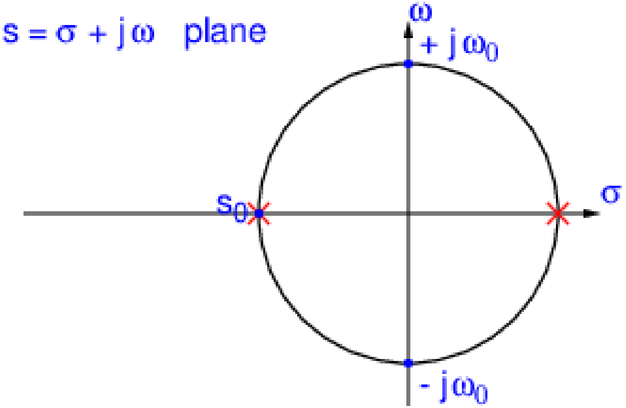
$$\phi_k = \frac{\pi}{2} + \frac{(k+1/2)\pi}{n}$$

is the angle of s_k measured anticlockwise from the positive real axis.

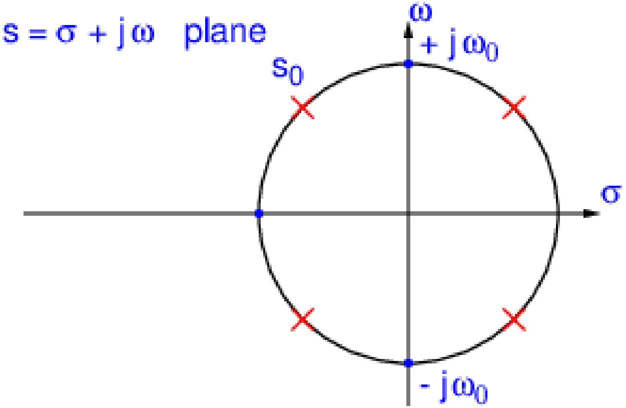
Butterworth LPF: Poles of $T(s)T(-s)$

- $|s_k| = \omega_c$. So all the poles of $T(s)T(-s)$ lie on a circle of radius ω_c.
- Increasing k by $2n$ increases ϕ_k by 2π. So there are only $2n$ distinct poles. So we shall use index k values from 0 to $2n-1$.
- Increasing k by 1 increases ϕ_k by π/n. So the poles are equispaced on the circle of radius ω_c, with the angle between two adjacent poles being π/n.
- k values from 0 to $n-1$ give ϕ_k values between $\pi/2$ and $3\pi/2$. So these poles lie in the left half plane (LHP). These are to be taken as the poles of $T(s)$ to satisfy the stability requirement.
- Let us look at some example pole diagrams.

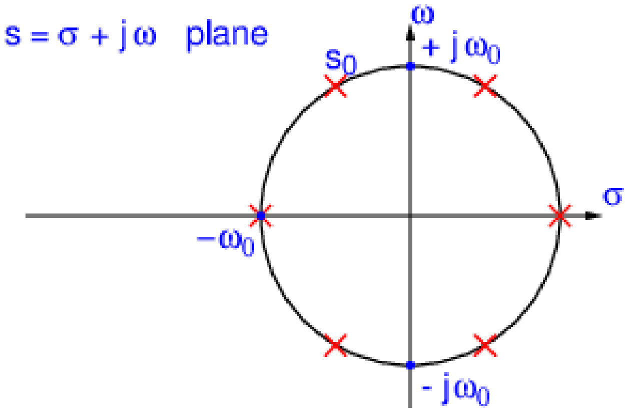
Butterworth LPF: Poles of $T(s)T(-s)$ for $n = 1$



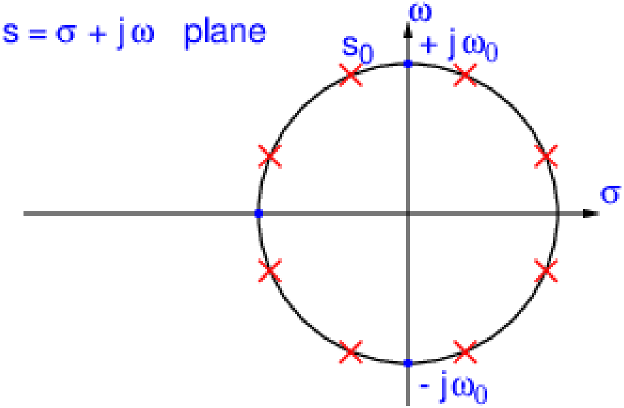
Butterworth LPF: Poles of $T(s)T(-s)$ for $n = 2$



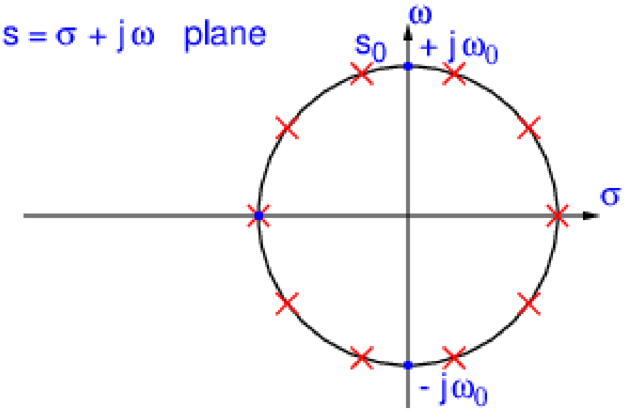
Butterworth LPF: Poles of $T(s)T(-s)$ for $n = 3$



Butterworth LPF: Poles of $T(s)T(-s)$ for $n = 4$



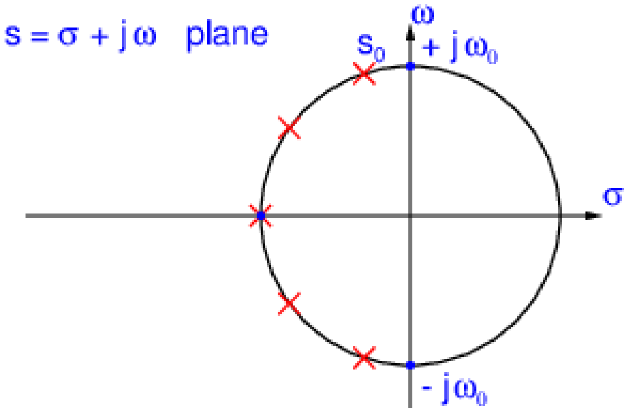
Butterworth LPF: Poles of $T(s)T(-s)$ for $n = 5$



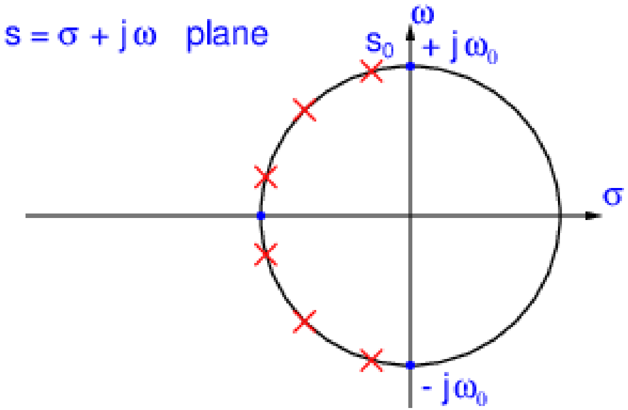
Butterworth LPF: Poles of $T(s)$

- Poles of $T(s)$ are to be taken as s_k for $k = 0, \ldots, n-1$, so that they lie in the LHP.
- We have n poles.
- Complex conjugates: $\overline{s_k} = s_{n-1-k}$.
  This is because $\phi_{n-1-k} + \phi_k = 2\pi$.
- If n is odd, then $s_{(n-1)/2} = -\omega_c$.
- For n even, we have $n/2$ complex conjugate pairs of poles.
  These can be implemented by $n/2$ Sallen-Key LPF sections.
- For n odd, we have $(n-1)/2$ complex conjugate pairs of poles, and one real pole at $-\omega_c$.
  These can be implemented by $(n-1)/2$ Sallen-Key LPF sections, and a first-order RC LPF section.

Butterworth LPF: Poles of $T(s)$ for $n = 5$



Butterworth LPF: Poles of $T(s)$ for $n = 6$



Butterworth LPF: List of Sections

n even:

- $n/2$ Sallen-Key sections with k going from 0 to $n/2 - 1$.

n odd:

- $(n - 1)/2$ Sallen-Key sections with k going from 0 to $(n - 1)/2 - 1$.
- One first-order RC LPF section.

Note that in each case, the index k begins with 0.

Butterworth LPF: Sallen-Key Section Design

Since

$$\phi_k = \frac{\pi}{2} + \frac{(k+1/2)\pi}{n}$$

$$\psi_k = \pi - \phi_k = \frac{\pi}{2} - \frac{(k+1/2)\pi}{n}$$

$$Q_k = \frac{1}{2\cos\psi_k} = \frac{1}{2\sin\frac{(k+1/2)\pi}{n}}$$

All ω_0 values are equal to ω_c.

$$\omega_{0,k} = \omega_c$$

All H values are equal to 1.

$$H_k = 1$$

Butterworth LPF Demonstration

- On web page, check for various *n*.
- Run **ngspice** for one output.