Generic quantum walk using a coin-embedded shift operator

C. M. Chandrashekar

Institute for Quantum Computing, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1
and Perimeter Institute for Theoretical Physics, Waterloo, Ontario, Canada N2J 2W9

The study of quantum walk processes has been widely divided into two standard variants, the discrete-time quantum walk (DTQW) and the continuous-time quantum walk (CTQW). The connection between the two variants has been established by considering the limiting value of the coin operation parameter in the DTQW, and the coin degree of freedom was shown to be unnecessary [F. W. Strauch, Phys. Rev. A **74**, 030301(R) (2006)]. But the coin degree of freedom is an additional resource which can be exploited to control the dynamics of the QW process. In this paper we present a generic quantum walk model using a quantum coin-embedded unitary shift operation U_C. The standard version of the DTQW and the CTQW can be conveniently retrieved from this generic model, retaining the features of the coin degree of freedom in both variants.

I. INTRODUCTION

The quantum walk (QW) as it is known today is a generalization of the classical random walk (CRW) developed by exploiting the aspects of quantum mechanics, such as superposition and interference [1–5]. In the CRW the particle moves in the position space with a certain probability, whereas the QW, which involves a superposition of states, moves by exploring multiple possible paths simultaneously with the amplitudes corresponding to different paths interfering. This makes the variance of the QW on a line grow quadratically with the number of steps, compared to the linear growth for the CRW. A probabilistic result is obtained upon measurement. Several quantum algorithms have been proposed using QWs [6–9]. Experimental implementation of the QW has been reported [10–12], and various other schemes have been proposed for its physical realization [13–17]. Beyond quantum computation, they can be used to demonstrate coherent quantum control over atoms, photons, or spin chains. The quantum phase transition using a QW is one of them [18]. Direct experimental evidence for wavelike energy transfer within photosynthetic systems has been reported, emphasizing the role of the QW [19].

There are two widely studied variants of the QW, the continuous-time quantum walk (CTQW) and the discrete-time quantum walk (DTQW) [20]. In the CTQW [21], one can directly define the walk on the position space whereas, in the DTQW [22], it is necessary to introduce a quantum coin operation to define the direction in which the particle has to move. The results from the CTQW and the DTQW are often similar, but due to the coin degree of freedom the discrete-time variant has been shown to be more powerful than the other in some contexts [9], and the coin parameters can be varied to control the dynamics of the evolution [4,23]. To match the performance of a spatial search using the DTQW, the coin degree of freedom has been introduced in the CTQW model [24]. The relation between the DTQW and the CTQW remained unclear and was an open problem [25] until the limiting value of the coin operation parameter in the DTQW was considered to establish the connection [26]. Later, the Dirac equation–DTQW–CTQW relationship was

also established [27]. In this construction the coin degree of freedom was shown to be unnecessary. But a coin degree of freedom is an extra resource; the parameters of the quantum coin can be exploited to control the evolution of the QW with potential applications in quantum computation [23] and to simulate and control the dynamics in physical systems [18]. The previous closest connection between the DTQW and CTQW is the weak limit theorem for the probability density [28–30].

The main motivation for this paper is to construct a generic QW model that will retain the features of the coin operation and establish the connection between the standard variants of the QW. Since the QW is a quantization of the classical diffusion process, it is quite natural to think in the direction of a generic model which leads to the different known variants of the QW under restrictions on the degrees of freedom of the physical system or the external resources used for implementing the QW.

We construct a generic model as an extension of the DTQW model. We replace the fixed local unitary shift operator U by the fixed local *coin-embedded shift operator* U_C. This will eliminate a separate coin toss operation on the particle to define the direction of the motion but retains the features of the coin operation. U_C is a physically feasible construction which will reduce the generic model to the standard version of the DTQW or the CTQW depending on the restriction on the degrees of freedom of the initial physical system. It is well known that physical systems are not free of environmental effects and it is shown that QW behavior is very sensitive to environmental effects [31–33]. The environmental effects on the two operations, the coin operation C and the unitary shift operation U, used in the realized and most of the proposed implementable schemes of the DTQW contribute to a decrease in the decoherence time of the system. The single operation U_C in the generic model replaces the two operations C and U. This reduction to a single operation effectively contributes to an increase in the decoherence time, which in turn contributes to the increase in the number of implementable steps in the given system. The single operation U_C also retains the features of quantum coin parameters.

In Sec. II we briefly describe the standard variants of the CTQW and the DTQW. Section III discusses the construction of the generic QW model. In Secs. IV A and IV B, the conditions to retrieve the standard versions of the DTQW and the CTQW are presented. With a brief description of the physical implementation in Sec. V, we conclude in Sec. VI.

II. THE TWO VARIANTS OF THE QW

We will recall both the standard variants of the quantum walk in this section. The review article by Kempe [20] discusses them in detail. In the CTQW [21], the walk is defined on the *position* Hilbert space $\mathcal{H}_p$ spanned by the basis state $|\psi_x\rangle$, $x \in \mathbb{Z}$. To implement the CTQW, the Hamiltonian H is defined such that

$$H|\psi_x\rangle = -|\psi_{x-1}\rangle + 2|\psi_x\rangle - |\psi_{x+1}\rangle \tag{1}$$

and is made to evolve with time t by applying the transformation

$$U(t) = \exp(iHt). \tag{2}$$

The Hamiltonian H of the process acts as the generator matrix which will transform the probability amplitude at the rate of γ to the neighboring sites. γ is a fixed, time-independent constant.

The one-dimensional DTQW [22] is defined on the Hilbert space $\mathcal{H} = \mathcal{H}_c \otimes \mathcal{H}_p$, where $\mathcal{H}_c$ is the *coin* Hilbert space spanned by the basis states of the particle, $|0\rangle$ and $|1\rangle$. To implement the DTQW, the quantum coin toss operation C, which in general can be an arbitrary U(2) operator [4], is applied on the particle at the origin in the state

$$|\psi_{\text{in}}\rangle = [\cos(\delta)|0\rangle + e^{i\eta}\sin(\delta)|1\rangle] \otimes |\psi_0\rangle. \tag{3}$$

For the description we will consider an arbitrary three-parameter SU(2) operator of the form

$$C_{\xi,\theta,\zeta} \equiv \begin{pmatrix} e^{i\xi}\cos(\theta) & e^{i\zeta}\sin(\theta) \\ e^{-i\zeta}\sin(\theta) & -e^{-i\xi}\cos(\theta) \end{pmatrix} \tag{4}$$

to get additional control over the evolution. The quantum coin operation $C_{\xi,\theta,\zeta}$ is followed by the conditional unitary shift operation

$$U = \exp(-2i\sigma_z \otimes Pl), \tag{5}$$

where P is the momentum operator and σ_z the Pauli z operator corresponding to a step of length l. The eigenstates of σ_z are denoted by $|0\rangle$ and $|1\rangle$. Therefore, U in the form of the state of the particle takes the form

$$U = |0\rangle\langle 0| \otimes \sum_{x \in \mathbb{Z}} |\psi_{x-1}\rangle\langle\psi_x| + |1\rangle\langle 1| \otimes \sum_{x \in \mathbb{Z}} |\psi_{x+1}\rangle\langle\psi_x|. \tag{6}$$

The process of

$$W_{\xi,\theta,\zeta} = U(C_{\xi,\theta,\zeta} \otimes \mathbb{1}) \tag{7}$$

is iterated without resorting to an intermediate measurement to realize a large number of steps of the QW. The three variable parameters of the quantum coin, ξ, θ, and ζ, can be varied to change the probability amplitude distribution in the
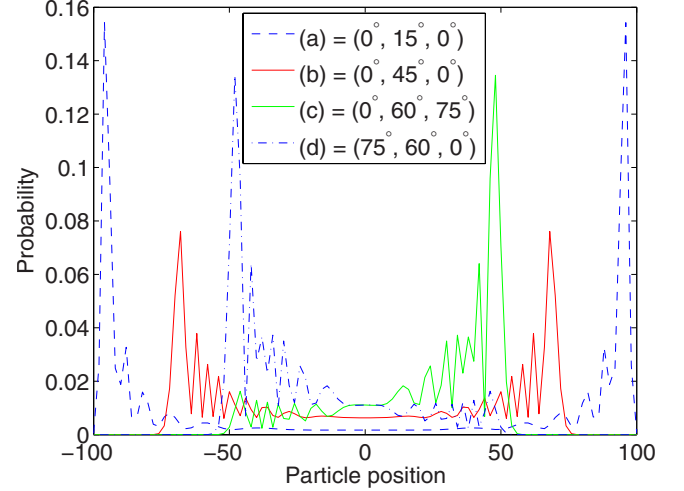

FIG. 1. (Color online) Distribution of the 100-step DTQW. The spread of the probability distribution for different values of θ using the operator $U_{0,\theta,0}$, is wider for (a)$=(0,\pi/12,0)$ than for (b) $=(0,\pi/4,0)$. Biasing the walk using ζ shifts the distribution to the right, (c)$=(0,\pi/3,5\pi/12)$ and ξ shifts it to the left, (d) $=(5\pi/12,\pi/3,0)$. The initial state of the particle $|\Psi_{\text{in}}\rangle = 1/\sqrt{2}(|0\rangle + i|1\rangle) \otimes |\psi_0\rangle$.

position space, Fig. 1. δ and η can be varied to get different initial states of the particle. By varying the parameter θ the variance can be increased or decreased via the functional form

$$\sigma^2 \approx [1 - \sin(\theta)]N^2. \tag{8}$$

For a particle with a symmetric superposition as the initial state, the parameters ξ and ζ introduce asymmetry in the probability distribution and their effect on the variance is very small. For a particle with an asymmetric superposition as the initial state, the parameters ξ and ζ can be configured to obtain a symmetric probability distribution [23].

III. GENERIC QW USING THE COIN-EMBEDDED SHIFT OPERATOR

The generic QW model is constructed as an extension of the standard version of the DTQW. In the standard DTQW model, $\mathcal{H}_c$ is spanned by the basis state of the particle, $|0\rangle$ and $|1\rangle$, whereas for the generic model we will introduce an additional degree of freedom,

$$\mathcal{H}_c = \mathcal{H}_{c_1} \otimes \mathcal{H}_{c_2}. \tag{9}$$

$\mathcal{H}_{c_1}$ is spanned by the basis states $|0_U\rangle$ and $|1_U\rangle$ of the external resource which is used to implement a coin-embedded unitary displacement U_C, and $\mathcal{H}_{c_2}$ is spanned by the basis states of the particle. Depending on the state of the external resource and the state of the particle, the U_C will implement the QW, eliminating the need for a separate coin toss operation after every unitary displacement.

To construct U_C, first let us consider the unitary shift operation used in the DTQW model, Eq. (5), which can also take the form

$$U = e^{-2i\sigma_z \otimes Pl} = e^{-i(|0\rangle\langle 0| - |1\rangle\langle 1|)\otimes Pl}$$

$$= (|0\rangle\langle 0| \otimes e^{-iPl})(|1\rangle\langle 1| \otimes e^{iPl}). \quad (10)$$

To embed a coin operation into the above expression, the external resource that is used to implement the unitary shift operation on the particle has to be defined such that it is local, that is, at each and every position space it is in the superposition state

$$|\Psi_U\rangle = [\cos(\theta)|0_U\rangle + e^{i\gamma}\sin(\theta)|1_U\rangle]. \quad (11)$$

If the external resource that implements the shift operator is in the state $|0_U\rangle$, then the particle in state $|0\rangle$ shifts to the left and the particle in state $|1\rangle$ shifts to the right. If the external resource is in the state $|1_U\rangle$ then the particle in state $|0\rangle$ shifts to the right and the particle in state $|1\rangle$ shifts to the left.

From the above description, the coin-embedded shift operation U_C takes the form

$$U_C = \{|0_U\rangle\langle 0_U| \otimes \exp[-i(|0\rangle\langle 0| - |1\rangle\langle 1|) \otimes Pl]\}$$

$$\times \{|1_U\rangle\langle 1_U| \otimes \exp[i(|0\rangle\langle 0| - |1\rangle\langle 1|) \otimes Pl]\}. \quad (12)$$

Therefore, U_C in the form of the state of the external resource and the particle can be written as

$$U_C = |0_U\rangle\langle 0_U| \otimes \left(|0\rangle\langle 0| \otimes \sum_{x\in\mathbb{Z}} |\psi_{x-1}\rangle\langle\psi_x| \right.$$

$$+ |1\rangle\langle 1| \otimes \sum_{x\in\mathbb{Z}} |\psi_{x+1}\rangle\langle\psi_x| \Bigg)$$

$$+ |1_U\rangle\langle 1_U| \otimes \left(|0\rangle\langle 0| \otimes \sum_{x\in\mathbb{Z}} |\psi_{x+1}\rangle\langle\psi_x| \right.$$

$$+ |1\rangle\langle 1| \otimes \sum_{x\in\mathbb{Z}} |\psi_{x-1}\rangle\langle\psi_x| \Bigg). \quad (13)$$

The operation U_C on the initial state of the system is of the form

$$|\Psi_{in}\rangle = |\Psi_U\rangle \otimes |\psi_p\rangle \otimes |\psi_0\rangle$$

$$= [\cos(\theta)|0_U\rangle + e^{i\gamma}\sin(\theta)|1_U\rangle]$$

$$\otimes [\cos(\delta)|0\rangle + e^{i\eta}\sin(\delta)|1\rangle] \otimes |\psi_0\rangle \quad (14)$$

and implements the first step of the generic QW; here $|\psi_p\rangle \otimes |\psi_0\rangle$ is the state of the particle at the origin (position). Hereafter, we will write the state of the particle position after t steps as $|\Psi_t\rangle$. Since $|\Psi_U\rangle$ is a local state of the external resource, after implementing U_C, the state of the particle position unentangles from the external resource to again entangle with the resource state $|\Psi_U\rangle$ in the new position. This can be written as

$$(1 \otimes |\Psi_t\rangle) = U_C(|\Psi_U\rangle \otimes 1)(1 \otimes |\Psi_{t-1}\rangle). \quad (15)$$

Therefore, irrespective of the internal state of the particle, U_C moves the particle in the superposition of the position space. Note that the role of the coin operation is completely retained in the above construction through the external resource which implements U_C. By choosing an equal superposition state of the external resource, Eq. (11), the distribution of the Hadamard walk can be retrieved.

If unit time is required to implement each step then to implement t steps $U_C(t) = U_C^t$. Therefore, the wave function after time t can be written as

$$(1 \otimes |\Psi_t\rangle) = [U_C(|\Psi_U\rangle \otimes 1)]^t(1 \otimes |\Psi_0\rangle). \quad (16)$$

The probability of the particle being in position x is

$$P_x(t) = |\langle x|\psi_{x,t}\rangle|^2. \quad (17)$$

By choosing a different linear combination of the initial state of the particle and the external resource implementing U_C, the probability distribution in the position space can be controlled, as is done using separate coin operations in the DTQW model.

IV. RETRIEVING THE STANDARD VERSIONS FROM THE GENERIC MODEL

A. DTQW

If the external resource implementing U_C is not in a superposition, that is, if it is in one of its basis states $|0_C\rangle$ or $|1_C\rangle$, then the Hilbert space, Eq. (9), $\mathcal{H}_c \equiv \mathcal{H}_{c_1}$. Therefore U_C, Eq. (12), reduces to

$$U_C = (|0\rangle\langle 0| \otimes e^{\mp iPl})(|1\rangle\langle 1| \otimes e^{\pm iPl}), \quad (18)$$

a unitary shift operator of the standard version of the DTQW, Eq. (6). Therefore, by introducing the quantum coin operation $C_{\xi,\theta,\zeta}$, the standard version of the DTQW and all its properties can be recovered.

B. CTQW

If the initial state of a particle is not in a superposition, that is, if it is in only one of its basis states $|0\rangle$ or $|1\rangle$, then the Hilbert space, Eq. (9), $\mathcal{H}_c \equiv \mathcal{H}_{c_2}$. Therefore U_C, Eq. (12), reduces to

$$U_C = (|0_U\rangle\langle 0_U| \otimes e^{\mp iPl})(|1_U\rangle\langle 1_U| \otimes e^{\pm iPl}), \quad (19)$$

$$U_C = \exp[\mp i(|0_U\rangle\langle 0_U| - |1_U\rangle\langle 1_U|) \otimes Pl]. \quad (20)$$

If it takes unit time for each U_C operation then after time t $U_C(t)$ can be written as

$$U_C^t = \exp[\mp i(|0_U\rangle\langle 0_U| - |1_U\rangle\langle 1_U|) \otimes Plt]. \quad (21)$$

Since $|0_U\rangle$ and $|1_U\rangle$ are the states of the external resource used to displace the particle, the above expression reveals the effect of the state of the external resource on the particle,

$$[(|0_U\rangle\langle 0_U| - |1_U\rangle\langle 1_U|) \otimes Pl](|\Psi_U\rangle \otimes \Psi_0\rangle)$$

$$= \alpha|0_U\rangle \otimes |\Psi_{-1}\rangle + \beta|1_U\rangle \otimes |\Psi_{+1}\rangle, \quad (22)$$

where α and β are the coefficients of the states $|0_U\rangle$ and $|1_U\rangle$ and $|\Psi_0\rangle$ is the state of the particle in postion 0. Since the external resource is a local state, Eq. (22) is

$$\equiv \alpha|\Psi_{-1}\rangle + \beta|\Psi_{+1}\rangle. \quad (23)$$

Therefore, Eq. (21) can be written in the form

$$U_C^t \equiv \exp(\pm iH_Lt), \quad (24)$$

where H_L is the local Hamiltonian in the position space. The probability amplitude transition rate γ is related to the state

of the external resource. By choosing an arbitrary superposition state of the external resource, Eq. (11), different transition rates, γ_1 for the left and γ_2 for the right can be obtained; this is also equivalent to the introduction of the coin degree of freedom to the standard CTQW model [24]. Thus the generic quantum walk model can be reduced to the standard version of the CTQW. The CTQW happens irrespective of the state of the particle. All the features of the coin operation in the standard version of the DTQW can be retrieved in this version of the CTQW. This construction also makes the connection between the DTQW and the CTQW very straightforward.

V. PHYSICAL IMPLEMENTATION

A simple physical system can be considered in which the polarized light can act as a coin-embedded unitary shift operator U_C. It can be conditioned such that the vertically polarized light ($|0_U\rangle$) will shift the particle in state $|0\rangle$ to the left and the particle in state $|1\rangle$ to the right. Horizontally polarized light ($|1_U\rangle$) shifts the particle in state $|1\rangle$ to the left and the particle in state $|0\rangle$ to the right. Therefore light in a coherent superposition of the vertical and horizontal polarizations, $[\cos(\theta)|0_U\rangle + e^{i\gamma}\sin(\theta)|1_U\rangle]$, can implement the U_C on a particle. The other physical advantage of using the generic model for the implementation of the QW is the possibility of increasing the decoherence time by reducing the number of operations needed to implement each step of the QW. The environmental effects on the two operations, the coin operation C and the displacement operation U, in the physical

system contribute to a reduction in the decoherence time. Replacement of the two operations C and U by a single operation contributes to a decrease in the environmental effects on the system and increases its decoherence time.

VI. CONCLUSION

In summary, we have constructed a generic QW model by embedding the coin operation into the unitary shift operator U_C. The generic model retains the features of the coin operation and establishes the connection between the two standard versions of the DTQW and the CTQW. When the external resource that implements U_C is not in a superposition of its basis states, the standard version of the DTQW can be retrieved by introducing an addition coin operation. When the particle on which the generic model is implemented is not in a superposition of its internal states, the CTQW is retrieved along with the features of the coin degree of freedom. This makes the CTQW reproduce all the features of the standard version of the DTQW. This model, along with establishing a link between the two versions of the QW, can also play a prominent role in increasing the decoherence time of the system and hence in increasing the realizable number of steps in a given physical system.

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