

Zeno subspace in quantum-walk dynamics

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We investigate discrete-time quantum-walk evolution under the influence of periodic measurements in position subspace. The undisturbed survival probability of the particle at the position subspace $P(0,t)$ is compared with the survival probability after frequent (n) measurements at interval $\tau = t/n$, $P(0,\tau)^n$. We show that $P(0,\tau)^n > P(0,t)$ leads to the quantum Zeno effect in position subspace when a parameter θ in the quantum coin operations and frequency of measurements is greater than the critical value, $\theta > \theta_c$ and $n > n_c$. This Zeno effect in the subspace preserves the dynamics in coin Hilbert space of the walk dynamics and has the potential to play a significant role in quantum tasks such as preserving the quantum state of the particle at any particular position, and to understand the Zeno dynamics in a multidimensional system that is highly transient in nature.

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I. INTRODUCTION

In standard quantum theory, the time evolution of the state vector of the quantum system undergoes continuous unitary evolution until the system is measured. If very frequent measurements are performed on a quantum system, in order to ascertain whether it is still in its initial state, transitions to other states are hindered or boosted resulting in the quantum Zeno effect (QZE) or the inverse quantum Zeno effect (IZE), respectively [1–6]. The QZE is expected to occur widely in quantum systems. In particular, for time t with n measurements, the complete suppression of the transition to other states in the limit of $t/n \rightarrow 0$ is universal, common to all quantum systems; that is, the system is frozen to the initial state. However, in a multidimensional system, the QZE does not necessarily freeze everything. On the contrary, for frequent projections onto a multidimensional subspace, the system can evolve away from its initial state, although it remains in the subspace defined by the measurement. This continuing time evolution within the projected subspace has also been investigated [3,7–10]. In this paper, the quantum system we use to investigate the Zeno dynamics in the projected subspace is the discrete-time quantum-walk (DTQW) evolution in $2 \times K$ Hilbert space.

Quantum-walk (QW) evolution involves the quantum features of interference and superposition resulting in the quadratically faster spread in position space than its classical counterpart, classical random walk (CRW) [11–15] in one dimension. QWs are studied in two forms—continuous-time QW (CTQW) [15] and discrete-time QW (DTQW) [13,14,16,17]—and are found to be very useful from the perspective of quantum algorithms [18–21] (e.g., to demonstrate the coherent quantum control over atoms, quantum phase transition [22]; to explain phenomena such as the breakdown of an electric-field driven system [23] and direct experimental evidence for wavelike energy transfer within photosynthetic systems [24]; to generate entanglement between spatially separated systems [25]; and to induce Anderson localization of Bose-Einstein condensate in optical lattice [26]). On the experimental front, implementation of QWs with samples in an NMR system [27],

in the continuous tunneling of light fields through waveguide lattices [28], in the phase space of trapped ions [29], with single optically trapped neutral atoms [30], and with single photon [31] has been reported. Various other schemes have been proposed for their physical realization in different physical systems [32–34].

Unlike many quantum processes on which the QZE is widely studied, the DTQW is a controlled unitary evolution in which the constructive interference is directed away from the initial position $x = 0$. This reduces the amplitude of the particle at $x = 0$ to a very small value after the first few steps of the QW evolution (cf. recurrence nature of QW [35,36]), thus making the walk highly transient in nature. Introducing a decoherence channel to effectively mask the unitary evolution during each step of the DTQW decreases the transient behavior; therefore, the QZE can be shown by taking the rate of the measurement to ∞ (cf. Refs. [37,38], which discuss the QZE in CTQW). However, decoherence does not preserve the state subjected to QW evolution. In this paper we show that without introducing a decoherence channel, the unitary walk dynamics can be controlled to make it less transient by choosing the specific range of parameter in the quantum coin operation. Such a walk under the influence of periodic measurements in position subspace ($x = 0$) is shown to lead to the QZE preserving the state of the particle at that position. This yields a quantum Zeno subspace in which the dynamics in the coin Hilbert space \mathcal{H}_c of the walk is preserved. This observation can have implications for applications of QW to various quantum tasks such as preserving the quantum state [39] and quantum simulation of annealing processes [40].

This paper is arranged as follows. In Sec. II we describe the DTQW model on a line and its transient nature. In Sec. III we discuss the conditions of the walk dynamics leading to the Zeno effect in the position subspace. Finally, in Sec. IV we make our concluding remarks.

II. DISCRETE-TIME QUANTUM WALK AND ITS TRANSIENT NATURE

The DTQW in one dimension is modeled as a $2 \times K$ system, that is, a particle consisting of a two-level coin (a qubit) in the Hilbert space \mathcal{H}_c , spanned by $|0\rangle$ and $|1\rangle$, and K positions in the position Hilbert space \mathcal{H}_p , spanned by $|\psi_x\rangle$,

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where $x \in \mathbb{I}$, the set of integers. A t -step DTQW with unit time required for each step of walk is generated by iteratively applying a unitary operation W that acts on the Hilbert space $\mathcal{H}_c \otimes \mathcal{H}_p$,

$$|\Psi_t\rangle = W^t |\Psi_{\text{ins}}\rangle, \quad (1)$$

where $|\Psi_{\text{ins}}\rangle$ is the initial state of the particle at a particular position. We will choose a symmetric superposition state of the particle at position $x = 0$

$$|\Psi_{\text{ins}}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) \otimes |\psi_0\rangle \quad (2)$$

as the initial state throughout this paper. $W \equiv S(B \otimes \mathbb{1})$, where

$$B = B_{\xi, \theta, \zeta} \equiv \begin{pmatrix} e^{i\xi} \cos(\theta) & e^{i\zeta} \sin(\theta) \\ -e^{-i\zeta} \sin(\theta) & e^{-i\xi} \cos(\theta) \end{pmatrix} \in \text{SU}(2) \quad (3)$$

is the quantum coin operation. S is the controlled-shift operation

$$S \equiv \sum_x [|0\rangle\langle 0| \otimes |\psi_x - 1\rangle\langle \psi_x| + |1\rangle\langle 1| \otimes |\psi_x + 1\rangle\langle \psi_x|]. \quad (4)$$

The probability to find the particle at position x after t steps is given by

$$P(x, t) = \langle \psi_x | \text{tr}_c(|\Psi_t\rangle\langle \Psi_t|) | \psi_x \rangle. \quad (5)$$

For a walk on a particle with the initial state at the origin $|\Psi_{\text{ins}}\rangle$ using an unbiased coin operation, that is, $B_{0, \theta, 0} \equiv B_\theta$, the variance after t steps of walk is $[1 - \sin(\theta)]t^2$ and a symmetric probability distribution in position space is obtained [41]. In Fig. 1, the probability distribution of 50-step QW evolution for different values of θ in the quantum coin operation B_θ is shown. For $\theta = 0^\circ$, the two states $|0\rangle$ and $|1\rangle$ move away from each other ballistically without any interference effect. With an increase in θ the interference effect is seen, and the

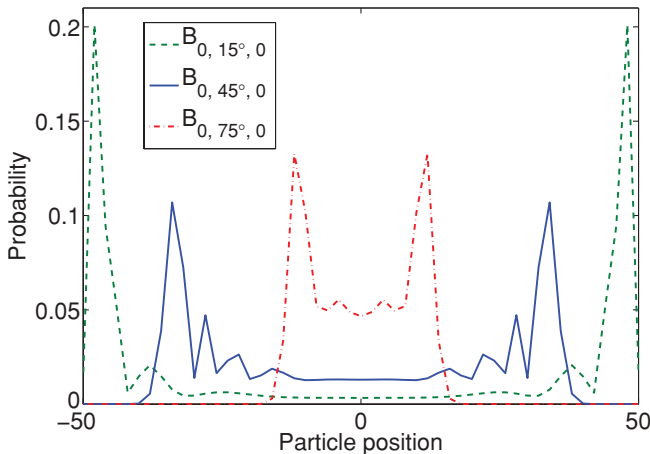


FIG. 1. (Color online) Spread of probability distribution of the QW evolution using different values of θ in the quantum coin operation B_θ . The distribution is wider for $\theta = 15^\circ$ and decreases with an increase in θ . The interference effect is absent for $\theta = 0^\circ$ and $\theta = 90^\circ$. The distribution is for 50 steps of walk and only even positions are plotted because the odd position will have a zero value for an even number of steps.

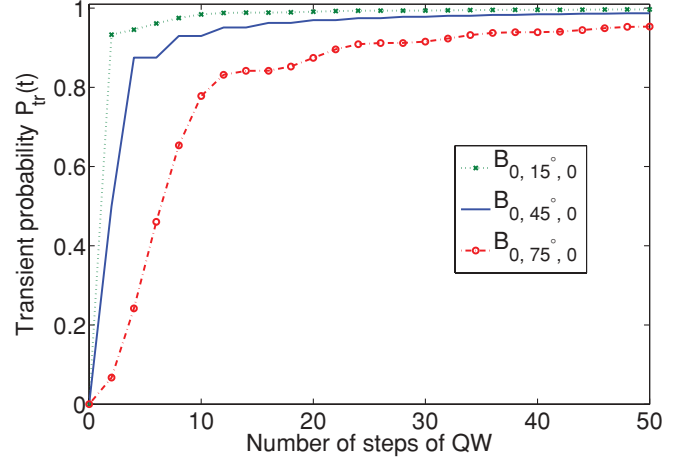


FIG. 2. (Color online) Transient probability $P_{\text{tr}}(t) = [1 - P(0, t)]$, probability of particle moving away from the initial position $x = 0$ with number of steps (time) for QW with $\theta = 15^\circ$, $\theta = 45^\circ$, and $\theta = 75^\circ$. We note that the $P_{\text{tr}}(t)$ shoots up very quickly for lower values of θ and, with an increase in θ , $P_{\text{tr}}(t)$ increases gradually.

distribution (which is wider for low values of θ) decreases with an increase in θ . The interference effect again disappears for another extreme value of $\theta = 90^\circ$. The two horned peaks on either side of the distribution, which moves away with an increase in the number of steps, makes QW highly transient in nature.

In Fig. 2 we have plotted the transient probability

$$P_{\text{tr}}(t) = [1 - P(0, t)], \quad (6)$$

that is, the probability of the particle moving away from the initial position $x = 0$ with the number of steps (time). For lower values of θ , the $P_{\text{tr}}(t)$ shoots up very quickly, and with an increase in θ , $P_{\text{tr}}(t)$ increases gradually. Therefore, making a measurement at position $x = 0$ for large values of θ will yield a survival probability. This behavior is the key for us to explore the quantum Zeno region (QZR) in the DTQW evolution.

III. ZENO EFFECT IN SUBSPACE OF DISCRETE-TIME QUANTUM WALK

In this section we outline the conditions for performing the measurements and using quantum coin parameters to observe the QZE in the subspace of the walk dynamics, preserving the dynamics in the coin Hilbert space \mathcal{H}_c . We first consider the position $x = 0$ as the subspace $\mathcal{H}_p^s \in \mathcal{H}_p$ from the complete Hilbert space of the DTQW system $\mathcal{H} = \mathcal{H}_c \otimes \mathcal{H}_p$ to study the QZE. One of the most trivial ways to freeze the particle at subspace \mathcal{H}_p^s with $P(0, t) = 1$, resulting in the QZE, is by making projective measurements in \mathcal{H}_p^s at intervals far less than the time required to implement one step of the walk ($\tau \ll 1$, with unit time required for each step). Due to the transient nature of the DTQW, for $\tau \geq 1$, $P(0, \tau) \neq 1$, observing the Zeno effect is not straightforward. However, by being selective in performing the measurements, we can see the QZR [42] if the undisturbed survival probability of the particle in \mathcal{H}_p^s is less than the survival probability with n measurements, that is,

$$P(0, t) < P(0, \tau)^n, \quad (7)$$

where $t = n\tau$. Measurements have to be selective for DTQW evolution because, for every odd number of steps of walk, the probability at subspace \mathcal{H}_p^s and other even positions is always zero. Therefore, intervals we need to consider for measurements are $t = 2$ and its multiples. If we consider the measurement after the first two steps of the walk, the state only at subspace \mathcal{H}_p^s is retained and the rest is discarded, the QW is further evolved for the next two steps, and the process is repeated many times before calculating the survival probability.

It is convenient to discuss the evolution of t -step walk using $W \equiv S(B \otimes \mathbb{1})$ in terms of the density matrix $\rho(x, t)$. If $\rho(0, 0) = |\Psi_{\text{ins}}\rangle\langle\Psi_{\text{ins}}|$, the density matrix after t steps of walk will be

$$\rho(x, t) = (W^t)\rho(0, 0)(W^t)^\dagger. \quad (8)$$

By taking projective measurements on the position subspace $|\psi_0\rangle$, we get

$$\rho(0, t) = \langle\psi_0|(W^t)\rho(0, 0)(W^t)^\dagger|\psi_0\rangle. \quad (9)$$

Then the survival probability at subspace \mathcal{H}_p^s after a single measurement is

$$P(0, t) = \text{tr}_c[\rho(0, t)]. \quad (10)$$

If the projective measurement on the position subspace \mathcal{H}_p^s is made after the first two steps of the walk, the state $|\Psi(0, 2)\rangle$ will be

$$|\Psi(0, 2)\rangle = [-e^{i(\xi-\zeta)} \cos(\theta) \sin(\theta) - i \sin^2(\theta)]|0\rangle + [-\sin^2(\theta) + i e^{-i(\xi-\zeta)} \cos(\theta) \sin(\theta)]|1\rangle. \quad (11)$$

In the preceding expression, we can note that the survival probability is largely dependent on θ and, due to symmetric contributions from the neighboring lattice to the position $x = 0$, one can ignore the roles of ξ and ζ in the survival probability of the state. The general form of the two-component vector of amplitudes of the particle, at position x and at time t , with left-moving (L) and right-moving (R) components, can be written in the form

$$\begin{pmatrix} \Psi_L(x, t) \\ \Psi_R(x, t) \end{pmatrix} = \begin{pmatrix} e^{i\xi} \cos(\theta) \Psi_L(x+1, t-1) \\ +e^{i\xi} \sin(\theta) \Psi_R(x-1, t-1) \\ e^{-i\xi} \cos(\theta) \Psi_R(x-1, t-1) \\ -e^{-i\xi} \sin(\theta) \Psi_L(x+1, t-1) \end{pmatrix}. \quad (12)$$

For $x = 0$ and any time t in the preceding expression [the argument used for evolution with measurement after two steps, Eq. (12)], the symmetric contribution from the neighboring lattice remains valid.

When n periodic measurements are made on the system,

$$P(0, \tau)^n = \text{tr}_c[(W_M^\tau)^n \rho(0, 0) (W_M^\tau)^{n\dagger}] \equiv \text{tr}_c[\rho(0, \tau)^n], \quad (13)$$

where W_M^τ is the unitary operation W with projective measurement onto \mathcal{H}_p^s after τ operations. For a Hadamard walk it is shown that $P(0, t) = O(t^{-1})$ [17] and for a walk with coin operation B , $P(0, t) = O(\frac{\sin(\theta)}{t})$. Therefore, $P(0, \tau)^n = O(\frac{\sin(\theta)}{\tau})^n$. For larger θ and small τ (i.e., more frequent measurements), we can see that $P(0, \tau)^n > P(0, t)$, leading to the QZR.

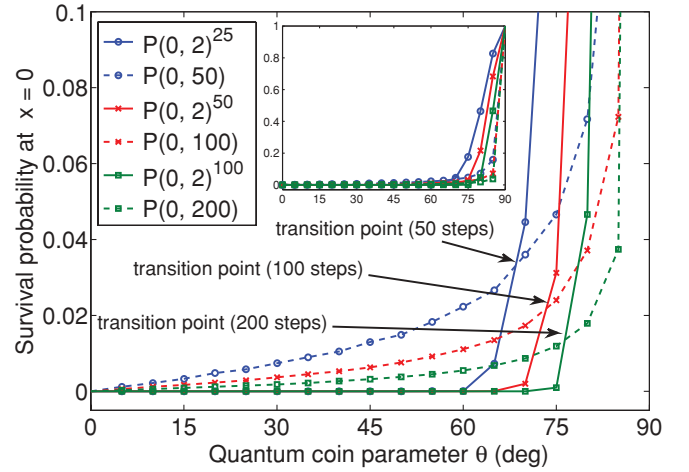


FIG. 3. (Color online) Closer view of the variation of the survival probability at position $x = 0$ with θ for 50-, 100-, and 200-step QW evolution with measurement after every two steps (solid lines) and after one single measurement at the end of the evolution (dashed lines). Transition of the dynamics to the quantum Zeno region $P(0, 2)^n > P(0, 2n)$ is observed at different θ for walks with different numbers of steps. The inset is a full plot of survival probability that becomes a unit value at $\theta = 90^\circ$.

In Fig. 3, undisturbed and disturbed survival probabilities of the state of the particle at position $x = 0$ for QW evolution with different θ in the quantum coin operation are shown for 50, 100, and 200 steps of walk. The undisturbed survival probability $P(0, t)$ is shown using dashed lines, and the disturbed survival probability with n measurements after every two steps, $P(0, 2)^n$, is shown using solid lines. For a QW with measurements after every two steps, $P(0, 2)^{25} \approx 0$ for $\theta < 60^\circ$ (for 50 steps), $P(0, 2)^{50} \approx 0$ for $\theta < 65^\circ$ (for 100 steps), and $P(0, 2)^{100} \approx 0$ for $\theta < 70^\circ$ (for 200 steps), respectively. At the transition point leading to the QZR, $P(0, \tau)^n = P(0, n\tau)$. Beyond $\theta > \theta_c$ we note that $P(0, 2)^n < P(0, 2n)$, leading the walk dynamics to the QZR. In Fig. 4, the survival probability with θ for 100-step QW evolution with different frequencies of measurements is shown. The QZR increases with an increase

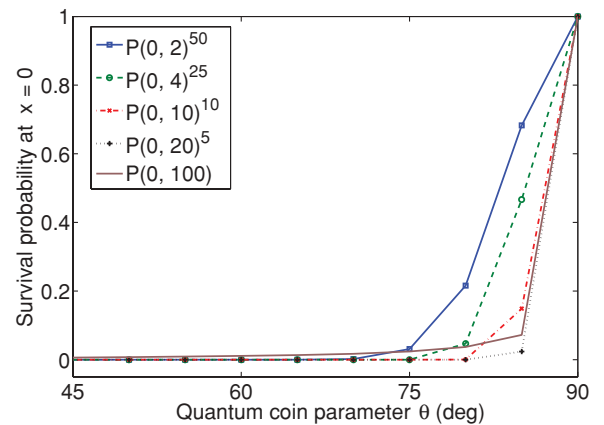


FIG. 4. (Color online) Variation of survival probability with θ for 100-step QW evolution with different frequencies of measurements. With an increase in the frequency of measurements, the QZR for the range of θ increases.

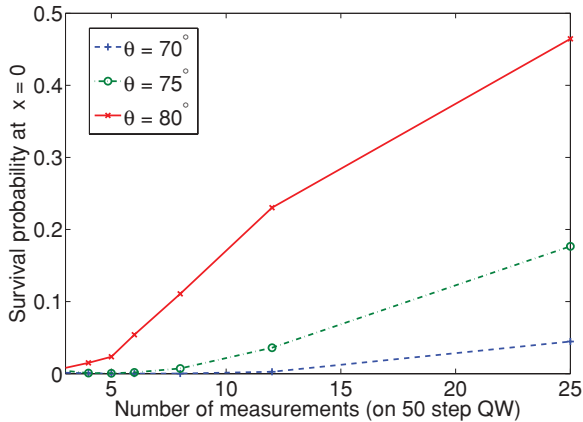


FIG. 5. (Color online) Variation of survival probability with number of measurements (n) for 50-step QW evolution with different θ . With an increase in the frequency of measurements and θ , the QZR increases. All three plots in the figure converge to a unit value for n (x axis) = 50.

in the frequency of measurements for a range of θ . In Fig. 5, the survival probability with number of measurements for 50-step QW evolution with different θ is shown. With an increase in the frequency of measurements and θ , the QZR also increases. For lower values of θ in Fig. 3, even though $P(0, \tau)^n < P(0, n\tau)$, we note that it does not continue to decrease (i.e., the opposite of Fig. 5) with an increase in the number of measurements. This suggests the absence of the inverse QZR.

IV. CONCLUDING REMARKS

We have discussed the DTQW evolution under the influence of the periodic measurements in position subspace that yields

Zeno subspace preserving the dynamics of the coin Hilbert space. The transient nature of the DTQW, which decreases with an increase in the value of parameter θ in the quantum coin operation, was used to explore the QZR. For particular values of θ and frequency of measurements, the transition from survival probability with measurements less than the undisturbed survival probability, to survival probability with measurements greater than the undisturbed survival probability $P(0, t) < P(0, \tau)^n$ when $t = n\tau$, was shown leading to the transition to the QZR. Because we did not consider the decoherence channel to suppress the walk dynamics leading to the Zeno effect, the dynamics of the state of the particle in the projected subspace was preserved. These observations can have implications for applications of the DTQW to various quantum tasks. In Ref. [39], an algorithm to preserve the quantum state was proposed making use of the QZE. Using the DTQW and the QZE with periodic measurements in the position subspace can also be used to preserve the quantum state of the particle, not only at $x = 0$ but also at any other particular position in the position space. This can be achieved by using the combination of undisturbed DTQW evolution, first to shift the peak of the QW to the desired location, followed by frequent measurements using the parameters that can result in the QZE. In Ref. [40], quantum simulation of the classical annealing system is proposed using the combination of the CTQW and the QZE in its dynamics. The ability to control the DTQW dynamics and the QZE using quantum operations can lead to further exploration of the annealing problem.

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