# Non-Markovian channel from the reduced dynamics of a coin in a quantum walk

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The quantum channels with memory, known as non-Markovian channels, are of crucial importance for a realistic description of a variety of physical systems, and pave ways for new methods of decoherence control by manipulating the properties of an environment such as its frequency spectrum. In this work, the reduced dynamics of a coin in a discrete-time quantum walk is characterized as a non-Markovian quantum channel. A general formalism is sketched to extract the Kraus operators for a $t$-step quantum walk. Non-Markovianity, in the sense of P indivisibility of the reduced coin dynamics, is inferred from the nonmonotonous behavior of distinguishability of two orthogonal states subjected to it. Furthermore, we study various quantum information-theoretic quantities of a qubit under the action of this channel, putting in perspective the role such channels can play in various quantum information processing tasks.

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## I. INTRODUCTION

The study of open quantum systems with memory has attracted a lot of attention over the last few years, since such systems describe a plethora of physical phenomena and also provide new ways to control various quantum features by engineering the system-environment interactions [1,2]. Several investigations on the role of structured environments and non-Markovianity in entanglement generation [3], quantum teleportation [4], key distribution [5], quantum metrology [6], and quantum biology [7] have suggested the advantage of non-Markovian quantum channels over Markovian ones.

The quantum walk (QW) was conceived as a generalization of classical random walks with an anticipation of its potential in modeling the dynamics particle in the quantum realm [8–13]. QWs describe the coherent evolution of a quantum particle, where the coin space is coupled to the position space which in principle can be treated as an external environment. One-dimensional QWs involve a walker free to move in either direction along a straight line such that the direction for each step is decided by the outcome of a coin toss. However, it differs from its classical counterpart in the sense that the probability distribution of the quantum particle spreads quadratically faster in position space than the classical random walk due to interference. This feature makes the QW an ideal candidate for development of quantum algorithms such as quantum search algorithms [14,15]. The ability to engineer the dynamics of the QWs has also allowed us to simulate and study quantum correlations [16–18], quantum-to-classical transition [19,20], memory effects and disorder [21], relativistic quantum effects [22], and quantum games [23]. Experimental implementation of QWs has also been realized in various physical systems, viz., in cold atoms [24,25] and photonic systems [26–32]. Studies have reported the circuit-based implementation of QWs [33–35]. A scheme for implementing QWs in Bose-Einstein condensates was presented in Ref. [36] and was recently implemented in momentum space [37]. Possible applications of QWs in understanding the dynamics in biological systems have been reported in various works [38–40], thus making QWs a topic of practical interest.

The QW can be discrete or continuous in time, accordingly known as a discrete-time quantum walk (DTQW) and continuous-time quantum walk (CTQW). In this work, we confine ourselves to the former case. The DTQW was studied from the perspective of various facets of non-Markovian evolution, such as the disambiguation of contributions to non-Markovian backflow as well as the transition from quantum to classical random walks [41]. The non-Markovian nature of coin dynamics in DTQWs can be brought out by tracing over the position space [42]. Henceforth, we use the term quantum walk noise (QWN) to describe the reduced dynamics on the coin space. In this work, we quantify this by developing the Kraus operators for the QWN, thereby characterizing the QW channel. The QWN was studied [41] in conjunction with a random telegraph noise (RTN) [43,44]. The P indivisibility [45–47] of the QWN as well as the RTN suggested that the intermediate map of the full evolution could be not completely positive (NCP). Also, nonmonotonic behavior under trace distance was indicated. This called for a careful consideration of the application of such non-Markovian noise channels to the DTQW protocol. A suggestion offered in Ref. [41] was that in contrast to the conventional application of the (Markovian) noise channel [19,20] in the form of appropriate Kraus operators [2], after each application of the walk operation, in the

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present non-Markovian scenario, the Kraus operators are applied once after $t$ QW steps. This notion was implemented numerically. Here, making use of the developed Kraus operators of the QW channel, we quantify this notion. It also serves the purpose of highlighting the implementation of non-Markovian noise channels to various QW protocols. We further characterize the QW channel by studying various information-theoretic processes on it. Specifically, the interplay of purity of the qubit state with the channel parameter as well as the state parameter is investigated. Furthermore, the Holevo quantity, which characterizes the information about an input state that can be retrieved from the output of the channel, is studied.

The paper is organized as follows: In Sec. II, the reduced coin dynamics is studied, sketching the formalism to extract the Kraus operators for a $t$-step walk. Section III is devoted to a detailed investigation of various properties of the QW channel, such as its non-Markovian nature in the sense of P indivisibility, the purity of states subjected to this channel, and the Holevo quantity. The conclusion of this work is presented in Sec. IV.

## II. REDUCED DYNAMICS OF A COIN

Let the initial states of coin and walker be $|\psi_c\rangle$ and $|\psi_p\rangle$, respectively. The unitary operator $\hat{W} = \hat{S}(\hat{C} \otimes \mathbb{1})$, where $\hat{S}$ and $\hat{C}$ are the shift and coin operators, respectively, governs the time evolution of the combined state $|\psi_c\rangle \otimes |\psi_p\rangle$. The state after $t$ steps is given by [48]

$$\begin{aligned} |\psi(t)\rangle &= \hat{W}^t(|\psi_c\rangle \otimes |\psi_p\rangle) \quad \text{or} \\ \rho(t) &= \hat{W}^t(\rho_c \otimes \rho_p)(W^t)^\dagger. \end{aligned} \tag{1}$$

Here, $\rho(t) = |\psi(t)\rangle\langle\psi(t)|$, $\rho_p = |\psi_p\rangle\langle\psi_p|$, and $\rho_c = |\psi_c\rangle\langle\psi_c|$ are the corresponding density matrices. Furthermore, the coin and shift operators are given by

$$\begin{aligned} \hat{C} &= \begin{pmatrix} \cos\theta & -i\sin\theta \\ -i\sin\theta & \cos\theta \end{pmatrix}, \\ \hat{S} &= |\uparrow\rangle\langle\uparrow| \otimes \hat{S}_L + |\downarrow\rangle\langle\downarrow| \otimes \hat{S}_R. \end{aligned} \tag{2}$$

The operators $\hat{S}_L = \sum_{x\in\mathbb{Z}} |x-1\rangle\langle x|$ and $\hat{S}_R = \sum_{x\in\mathbb{Z}} |x+1\rangle\langle x|$, are the left and right shift operators, respectively. The total unitary operator for $t$ steps becomes

$$\begin{aligned} W^t &= [\hat{S}(\hat{C} \otimes \mathbb{1})]^t \\ &= [(|\uparrow\rangle\langle\uparrow| \otimes \hat{S}_L + |\downarrow\rangle\langle\downarrow| \otimes \hat{S}_R)(\hat{C} \otimes \mathbb{1})]^t \\ &= [|\uparrow\rangle\langle\uparrow|\hat{C} \otimes \hat{S}_L + |\downarrow\rangle\langle\downarrow|\hat{C} \otimes \hat{S}_R]^t \\ &= [\hat{C}_\uparrow \otimes \hat{S}_L + \hat{C}_\downarrow \otimes \hat{S}_R]^t = [\hat{P} + \hat{Q}]^t. \end{aligned}$$

Here, $\hat{P} = \hat{C}_\uparrow \otimes \hat{S}_L$, $\hat{Q} = \hat{C}_\downarrow \otimes \hat{S}_R$, $\hat{C}_\uparrow = |\uparrow\rangle\langle\uparrow|\hat{C}$, and $\hat{C}_\downarrow = |\downarrow\rangle\langle\downarrow|\hat{C}$. The right-hand side can be simplified using the *binomial* expansion [49]

$$(\hat{P} + \hat{Q})^t = \sum_{k=0}^{t} \binom{t}{k} \hat{P}^k \hat{Q}^{t-k} + \sum_{k=0}^{t} \binom{t}{k} \hat{D}_k(\hat{Q}, \hat{P}) \hat{Q}^{t-k}. \tag{3}$$

The second term arises due to the noncommutative nature of $\hat{P}$ and $\hat{Q}$, and can be simplified using the recurrence relation,

$$\begin{aligned} \hat{D}_{k+1}(\hat{Q}, \hat{P}) &= [\hat{Q}, \hat{P}^k] + \hat{P}\hat{D}_k(\hat{Q}, \hat{P}) + [\hat{Q}, \hat{D}_k(\hat{Q}, \hat{P})], \\ &\text{with} \quad \hat{D}_0(\hat{Q}, \hat{P}) = 0. \end{aligned} \tag{4}$$

Thus, the quantity $\hat{D}_{k+1}(\hat{Q}, \hat{P})$ vanishes if $[\hat{Q}, \hat{P}] = 0$. From the definition of $\hat{P}$ and $\hat{Q}$, it follows that

$$[\hat{Q}, \hat{P}] = \hat{Q}\hat{P} - \hat{P}\hat{Q} = \hat{C}_\downarrow\hat{C}_\uparrow \otimes \mathbb{1} - \hat{C}_\uparrow\hat{C}_\downarrow \otimes \mathbb{1}. \tag{5}$$

Using the definition of $\hat{C}_{\uparrow(\downarrow)}$, it follows that

$$[\hat{Q}, \hat{P}] = \begin{pmatrix} -\sin^2\theta & -i\sin\theta\cos\theta \\ i\sin\theta\cos\theta & \sin^2\theta \end{pmatrix} \tag{6}$$

and is a zero matrix only for $\theta = 0, \pi$, and $2\pi$, which correspond to the coin operator being identity.

Further simplification of the first term in Eq. (3) reads

$$\begin{aligned} \sum_{k=0}^{t} \binom{t}{k} \hat{P}^k \hat{Q}^{t-k} &= \sum_k \binom{t}{k} (\hat{C}_\uparrow \otimes \hat{S}_L)^{t-k} (\hat{C}_\downarrow \otimes \hat{S}_R)^k \\ &= \sum_k \binom{t}{k} \hat{C}_\uparrow^{t-k} \hat{C}_\downarrow^k \otimes \hat{S}_L^{t-k} \hat{S}_R^k. \end{aligned} \tag{7}$$

For a walk of $t$ steps, symmetric about $x = 0$, the number of values a position can take is $2t + 1$. Let the initial states of coin and walker be $|\psi_c\rangle = a|\uparrow\rangle + b|\downarrow\rangle$ (with $|a|^2 + |b|^2 = 1$) and $|\psi_p\rangle = |x = 0\rangle$, respectively. The possible position states are $|x = -t\rangle, \ldots, |x = t\rangle$. We represent these states in computational basis as $(1\ 0\ 0\ldots)^T, \ldots, (0\ 0\ldots 1)^T$, respectively.

With this setting, we trace over the position degrees of freedom, using the notation $|x = \mu\rangle = |x_\mu\rangle$, and obtain

$$\begin{aligned} \rho_c(t) &= \sum_{\mu=-t}^{t} \langle x_\mu|\hat{W}^t(\rho_c \otimes |\psi_p\rangle\langle\psi_p|)(\hat{W}^t)^\dagger|x_\mu\rangle \\ &= \sum_{\mu=-t}^{t} K_\mu \rho_c K_\mu^\dagger. \end{aligned} \tag{8}$$

The Kraus operators are identified as

$$\begin{aligned} K_\mu &= \langle x_\mu|\hat{W}^t|\psi_p\rangle = \langle x_\mu|(\hat{P} + \hat{Q})^t|\psi_p\rangle \\ &= \sum_{k=0}^{t} \binom{t}{k} \langle x_\mu|\hat{P}^k \hat{Q}^{t-k}|\psi_p\rangle \\ &\quad + \sum_{k=0}^{t} \binom{t}{k} \langle x_\mu|\hat{D}_k(\hat{Q}, \hat{P})\hat{Q}^{t-k}|\psi_p\rangle, \end{aligned} \tag{9}$$

with $\mu = -t, \ldots, t$. In order to simplify the first term, we assume $|\psi_p\rangle = |0\rangle$; i.e., the walker starts at $x = 0$, such that

$$\begin{aligned} &\sum_{k=0}^{t} \binom{t}{k} \langle x_\mu|\hat{P}^k \hat{Q}^{t-k}|0\rangle \\ &= \sum_{k=0}^{t} \binom{t}{k} \langle x_\mu|(\hat{C}_\uparrow \otimes \hat{S}_L)^k (\hat{C}_\downarrow \otimes \hat{S}_R)^{t-k}|0\rangle \\ &= \sum_{k=0}^{t} \binom{t}{k} \hat{C}_\uparrow^k \hat{C}_\downarrow^{t-k} \langle x_\mu|\hat{S}_L^k \hat{S}_R^{t-k}|0\rangle \\ &= \sum_{k=0}^{t} \binom{t}{k} \hat{C}_\uparrow^k \hat{C}_\downarrow^{t-k} \delta_{\mu+k,t-k} \\ &= \frac{t!}{\left(\frac{t-\mu}{2}\right)!\left(\frac{t+\mu}{2}\right)!} \hat{C}_\uparrow^{\frac{t-\mu}{2}} \hat{C}_\downarrow^{\frac{t+\mu}{2}}. \end{aligned} \tag{10}$$

TABLE I. Kraus operators for the reduced coin dynamics for some steps of a symmetric QW. Here, $\theta$ is the coin parameter defined in Eq. (2).

| Steps | Kraus operators |
|---|---|
| 1 | $K_{-1}=\begin{pmatrix}0 & 0\\ -i\sin\theta & \cos\theta\end{pmatrix}$ $K_{1}=\begin{pmatrix}\cos\theta & -i\sin\theta\\ 0 & 0\end{pmatrix}$ |
| 2 | $K_{-2}=\begin{pmatrix}0 & 0\\ -i\cos\theta\sin\theta & \cos^2\theta\end{pmatrix}$ $K_{0}=\begin{pmatrix}-\sin^2\theta & -i\sin\theta\cos\theta\\ -i\sin\theta\cos\theta & -\sin^2\theta\end{pmatrix}$ $K_{2}=\begin{pmatrix}\cos^2\theta & -i\sin\theta\cos\theta\\ 0 & 0\end{pmatrix}$ |
| 3 | $K_{-3}=\begin{pmatrix}0 & 0\\ -i\cos^2\theta\sin\theta & \cos^3\theta\end{pmatrix}$ $K_{-1}=\begin{pmatrix}-\cos\theta\sin^2\theta & -i\cos^2\theta\sin\theta\\ -i\cos^2\theta\sin\theta+i\sin^3\theta & -2\cos\theta\sin^2\theta\end{pmatrix}$ $K_{1}=\begin{pmatrix}-2\cos\theta\sin^2\theta & -i\cos^2\theta\sin\theta+i\sin^3\theta\\ -i\cos^2\theta\sin\theta & -\cos\theta\sin^2\theta\end{pmatrix}$ $K_{3}=\begin{pmatrix}\cos^3\theta & -i\cos^2\theta\sin\theta\\ 0 & 0\end{pmatrix}$ |
| 4 | $K_{-4}=\begin{pmatrix}0 & 0\\ -i\cos^3\theta\sin\theta & \cos^4\theta\end{pmatrix}$ $K_{-2}=\begin{pmatrix}-\cos^2\theta\sin^2\theta & -i\cos^3\theta\sin\theta\\ -i\cos^3\theta\sin\theta+2i\cos\theta\sin^3\theta & -3\cos^2\theta\sin^2\theta\end{pmatrix}$ $K_{0}=\begin{pmatrix}-2\cos^2\theta\sin^2\theta+\sin^4\theta & -i\cos^3\theta\sin\theta+2i\cos\theta\sin^3\theta\\ -i\cos^3\theta\sin\theta+2i\cos\theta\sin^3\theta & -2\cos^2\theta\sin^2\theta+\sin^4\theta\end{pmatrix}$ $K_{2}=\begin{pmatrix}-3\cos^2\theta\sin^2\theta & -i\cos^3\theta\sin\theta+2i\cos\theta\sin^3\theta\\ -i\cos^3\theta\sin\theta & -\cos^2\theta\sin^2\theta\end{pmatrix}$ $K_{4}=\begin{pmatrix}\cos^4\theta & -i\cos^3\theta\sin\theta\\ 0 & 0\end{pmatrix}$ |

Use has been made of $\langle x_\mu|\hat{S}_L^k\hat{S}_R^{t-k}|0\rangle=\delta_{\mu+k,t-k}$ (see the Appendix). The constraints $k=(t-\mu)/2$ and $k\in\{0,1,2,\dots\}$ demand that $\mu$ and $t$ have same parity; i.e., for $t$ even (odd), $\mu$ is even (odd).

For a one-step walk, $t=1$ implies $\mu=-1,1$. From Eq. (4), $D_1(\hat{P},\hat{Q})=0$, and we have $K_\mu=\frac{t!}{(\frac{t-\mu}{2})!(\frac{t+\mu}{2})!}\hat{C}_\uparrow^{\frac{t-\mu}{2}}\hat{C}_\downarrow^{\frac{t+\mu}{2}}$, leading to

$$K_{-1}=\begin{pmatrix}0 & 0\\ -i\sin\theta & \cos\theta\end{pmatrix},\quad K_{1}=\begin{pmatrix}\cos\theta & -i\sin\theta\\ 0 & 0\end{pmatrix}. \tag{11}$$

These operators satisfy the completeness condition $K_{-1}^\dagger K_{-1}+K_1^\dagger K_1=\mathbb{1}$. Table I lists the Kraus operators for the reduced coin dynamics for a few steps of a symmetric QW. One infers the following:

(1) $K_{-t}=\mathcal{M}[K_t]$, where $\mathcal{M}[K_t]$ is the *minor* of the matrix $K_t$.

(2) For coin parameter $\theta=\pi/2$, $K_{\pm 2n}=0$, $n=1,2,3\dots$, and $K_0=\pm\mathbb{1}$, with $\mathbb{1}$ being the identity matrix.

The Kraus operators $K_t$ constitute a map $\mathcal{F}$ connecting the input state $\rho_c(0)$ to output $\rho_c(t)$. Let $\rho_c(0)=|\psi_c(0)\rangle\langle\psi_c(0)|$ with $|\psi_c(0)\rangle=a|\uparrow\rangle+b|\downarrow\rangle$, and we have

$$\rho_c(0)=\begin{pmatrix}|a|^2 & ab^*\\ a^*b & |b|^2\end{pmatrix}\to\rho_c(t)=[\mathcal{F}]_{t=n}\rho_c(0)$$
$$=\sum_{\mu=-t}^{t}K_\mu\rho_c(0)K_\mu^\dagger=\begin{pmatrix}p_t(\theta) & q_t(\theta)\\ q_t^*(\theta) & 1-p_t(\theta)\end{pmatrix}. \tag{12}$$

Here, $p_t(\theta)$ is the probability of obtaining $|\uparrow\rangle$ in a $t$-step walk. The form of $p_t(\theta)$ and $q_t(\theta)$ for some steps is given below:

$$\left.\begin{aligned}p_1(\theta)&=|a\cos\theta-b\sin\theta|^2\\ &=\tfrac{1}{2}[1+(|a|^2-|b|^2)\cos(2\theta)\\ &\quad+i(ab^*-a^*b)\sin(2\theta)],\\ p_2(\theta)&=\tfrac{1}{4}[1+2|a|^2+(|a|^2-|b|^2)\cos(4\theta)\\ &\quad+i(ab^*-a^*b)\sin(4\theta)],\\ p_3(\theta)&=\tfrac{1}{16}[6+4|a|^2+5(|a|^2-|b|^2)\cos(2\theta)\\ &\quad-2(|a|^2-|b|^2)\cos(4\theta)+3(|a|^2-|b|^2)\cos(6\theta)\\ &\quad+3i(ab^*-a^*b)\sin(2\theta)-2i(ab^*-a^*b)\sin(4\theta)\\ &\quad+3i(ab^*-a^*b)\sin(6\theta)],\end{aligned}\right\} \tag{13}$$

and

$$\left.\begin{aligned}q_1(\theta)&=0,\\ q_2(\theta)&=\sin^2\theta[ab^*\cos^2\theta+a^*b\sin^2\theta\\ &\quad+i(|a|^2-|b|^2)\sin\theta\cos\theta],\\ q_3(\theta)&=\cos\theta\sin^2\theta[(a^*b+ab^*)\cos\theta\\ &\quad+(ab^*-a^*b)\cos(3\theta)+i(|a|^2-|b|^2)\sin(3\theta)].\end{aligned}\right\} \tag{14}$$

The probabilities $p_t(\theta)$ are plotted in Figs. 1(a) and 1(b) when $|\uparrow\rangle=|0\rangle$, with respect to the coin parameter $\theta$. The asymmetric behavior of the probabilities, with respect to even and odd numbers of steps, is observed at $\theta=\pi/2$, where probabilities converge to one (zero) for even (odd) numbers of steps. The value of the coin parameter $\theta=\pi/2$ corresponds to the coin operator, Eq. (2), $\hat{C}=-i\sigma_x$, where $\sigma_x$ is the Pauli operator. This flips that state $|0\rangle$ ($|1\rangle$) to $|1\rangle$ ($|0\rangle$), thus returning

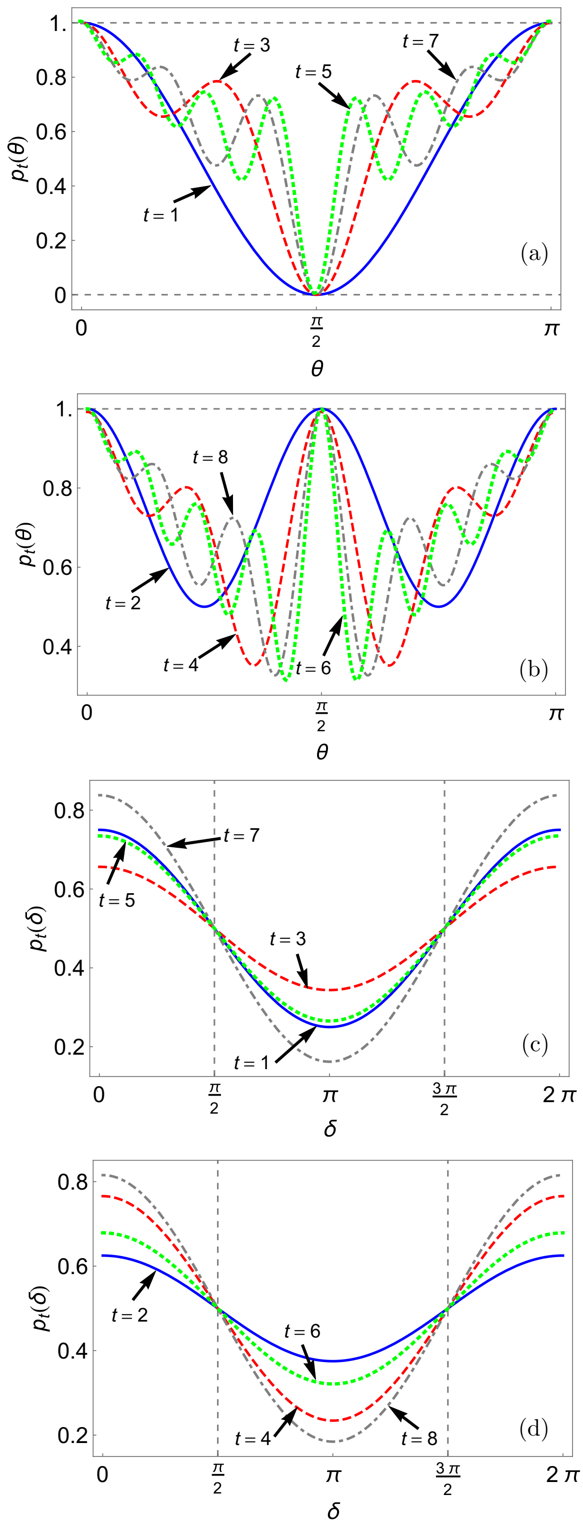


FIG. 1. Depicting probability $p_t$ [see Eq. (12)] of obtaining $|0\rangle$ in a $t$-step QW (a, b) with respect to the coin parameter $\theta$ with initial state $|\psi_c\rangle = |0\rangle$, and (c, d) with respect to the state parameter $\delta$ in with initial state $|\psi_c\rangle = \cos(\delta/2)\,|0\rangle + \sin(\delta/2)\,|1\rangle$, and coin parameter $\theta = \pi/6$.

probability one and zero of the initial state $|0\rangle$ after every even and odd step, respectively. For $\theta \neq \pi/2$ and $\neq 0$ interference in position space comes into play modifying the state and we see its effect on probability. In Figs. 1(c) and 1(d), the effect of the change in initial state is shown. The curve will flatten up (not shown but intuitive) around one for an even number of steps and around zero for an odd number of steps. There are other formulations of QW, like the *split-step* QW, where one breaks each step of the walk into two half-step evolutions described by the unitary $\hat{W}_{ss} = \hat{S}_+(\hat{C} \otimes \mathbb{1})\hat{S}_-(\hat{C} \otimes \mathbb{1}) = \hat{W}^2$ [50]. Here, $\hat{W}$ is the unitary operator for the standard QW, defined in Eq. (1). The Kraus operators for some steps of the split-step QW are given in Table II.

## III. SOME PROPERTIES OF THE QW CHANNEL

In this section, we characterize the non-Markovian QW channel comprising the reduced coin dynamics. We also study some quantum information-theoretic quantities on it.

### A. Non-Markovian dynamics

Non-Markovianity is a multifaceted phenomenon. Here, we restrict ourselves to the P-indivisibility form of non-Markovianity, which, for a map $\Phi_t = \Phi_{t,0}$ $(t > 0)$, means that the intermediate map $\Phi_{t,s} = \Phi_t \Phi_s^{-1}$, with $t > s > 0$, ceases to be positive [46]. A more general condition is when $\Phi_{t,s}$ is NCP, leading to CP-indivisible form of non-Markovian dynamics. The P-indivisible dynamics can be probed by using some state distinguishability measure, such as trace distance, denoted by $\mathcal{D}$. The trace distance of states $\rho$ and $\sigma$ is defined as $\mathcal{D}(\rho, \sigma) = \frac{1}{2}\sum_i |\lambda_i|$, where $\lambda_i$ are the eigenvalues of matrix $\rho - \sigma$. A departure from the monotonic behavior of $\mathcal{D}(\mathcal{A}(\rho), \mathcal{A}(\sigma))$ implies P indivisibility of the map $\mathcal{A}$, and hence non-Markovian dynamics. Consider two orthogonal states $\rho_0(t = 0) = |0\rangle\langle 0|$ and $\rho_1(t = 0) = |1\rangle\langle 1|$, subjected to the QW channel for a specific number of steps. For a one-step walk, we have

$$\mathcal{D}(\rho_0(n = 1), \rho_1(n = 1)) = \frac{1}{2}\sum_i |\lambda_i| = |\cos(2\theta)|. \quad (15)$$

Here, $\lambda_i$ are the eigenvalues of $\rho_0(n = 1) - \rho_1(n = 1)$ and

$$\rho_0(n = 1) = \sum_{\mu=1,3} K_\mu \rho_0 K_\mu^\dagger,$$
$$\rho_1(n = 1) = \sum_{\mu=1,3} K_\mu \rho_1 K_\mu^\dagger. \quad (16)$$

Similarly, we can compute the trace distance between $\rho_0(n)$ and $\rho_1(n)$ for an arbitrary $n$ number of steps, as depicted in Fig. 2(a). The fluctuating nature of the curves clearly brings out the P indivisibility of the non-Markovian QW channel comprising the reduced coin dynamics. The non-Markovian nature of the reduced dynamics of the coin can be attributed to the *entanglement* between coin and walker degrees of freedom and tracing over the subspace of the latter.

It is important to highlight the fact that for the case of non-Markovian processes, such as the P-indivisible case studied here, the concatenation of a one-step map $n$ times is not equivalent to operating with an $n$-step map, that is, $\mathcal{F}_1\mathcal{F}_1 \cdots \mathcal{F}_1 \neq \mathcal{F}_n$ (Fig. 3). This becomes clear when one computes the trace distance between $|0\rangle$ and $|1\rangle$, which turns out to be a

TABLE II. Kraus operators for the reduced coin dynamics for some steps in a split-step quantum walk.

| Steps | Kraus operators |
|---|---|
| 1 | $K_{-1}=\begin{pmatrix}\cos^2(\theta) & -i\cos(\theta)\sin(\theta)\\ 0 & 0\end{pmatrix}$ $K_0=\begin{pmatrix}-\sin^2(\theta) & -i\cos(\theta)\sin(\theta)\\ -i\cos(\theta)\sin(\theta) & -\sin^2(\theta)\end{pmatrix}$ $K_1=\begin{pmatrix}0 & 0\\ -i\cos(\theta)\sin(\theta) & \cos^2(\theta)\end{pmatrix}$ |
| 2 | $K_{-2}=\begin{pmatrix}-\cos^2(\theta)\sin^2(\theta) & -i\cos^3(\theta)\sin(\theta)\\ -\frac{1}{4}i(3\cos(2\theta)-1)\sin(2\theta) & -3\cos^2(\theta)\sin^2(\theta)\end{pmatrix}$ $K_{-1}=\begin{pmatrix}\cos^4(\theta) & -i\cos^3(\theta)\sin(\theta)\\ 0 & 0\end{pmatrix}$ $K_0=\begin{pmatrix}\sin^4(\theta)-2\cos^2(\theta)\sin^2(\theta) & -\frac{1}{4}i(3\cos(2\theta)-1)\sin(2\theta)\\ -\frac{1}{4}i(3\cos(2\theta)-1)\sin(2\theta) & \sin^4(\theta)-2\cos^2(\theta)\sin^2(\theta)\end{pmatrix}$ $K_1=\begin{pmatrix}0 & 0\\ -i\cos^3(\theta)\sin(\theta) & \cos^4(\theta)\end{pmatrix}$ $K_2=\begin{pmatrix}-3\cos^2(\theta)\sin^2(\theta) & -\frac{1}{4}i(3\cos(2\theta)-1)\sin(2\theta)\\ -i\cos^3(\theta)\sin(\theta) & -\cos^2(\theta)\sin^2(\theta)\end{pmatrix}$ |
| 3 | $K_{-3}=\begin{pmatrix}-5\cos^4(\theta)\sin^2(\theta) & -\frac{1}{2}i\cos^3(\theta)(5\cos(2\theta)-3)\sin(\theta)\\ -i\cos^5(\theta)\sin(\theta) & -\cos^4(\theta)\sin^2(\theta)\end{pmatrix}$ $K_{-2}=\begin{pmatrix}\frac{1}{8}(1-5\cos(2\theta))\sin^2(2\theta) & -\frac{1}{2}i\cos^3(\theta)(5\cos(2\theta)-3)\sin(\theta)\\ -\frac{1}{16}i(\sin(2\theta)-4\sin(4\theta)+5\sin(6\theta)) & \frac{1}{4}(1-5\cos(2\theta))\sin^2(2\theta)\end{pmatrix}$ $K_{-1}=\begin{pmatrix}\cos^6(\theta) & -i\cos^5(\theta)\sin(\theta)\\ 0 & 0\end{pmatrix}$ $K_0=\begin{pmatrix}-\frac{1}{4}(4\cos(2\theta)+5\cos(4\theta)+3)\sin^2(\theta) & -\frac{1}{16}i(\sin(2\theta)-4\sin(4\theta)+5\sin(6\theta))\\ -\frac{1}{16}i(\sin(2\theta)-4\sin(4\theta)+5\sin(6\theta)) & -\frac{1}{4}(4\cos(2\theta)+5\cos(4\theta)+3)\sin^2(\theta)\end{pmatrix}$ $K_1=\begin{pmatrix}0 & 0\\ -i\cos^5(\theta)\sin(\theta) & \cos^6(\theta)\end{pmatrix}$ $K_2=\begin{pmatrix}\frac{1}{4}(1-5\cos(2\theta))\sin^2(2\theta) & -\frac{1}{16}i(\sin(2\theta)-4\sin(4\theta)+5\sin(6\theta))\\ -\frac{1}{2}i\cos^3(\theta)(5\cos(2\theta)-3)\sin(\theta) & \frac{1}{8}(1-5\cos(2\theta))\sin^2(2\theta)\end{pmatrix}$ $K_3=\begin{pmatrix}-\cos^4(\theta)\sin^2(\theta) & -i\cos^5(\theta)\sin(\theta)\\ -\frac{1}{2}i\cos^3(\theta)(5\cos(2\theta)-3)\sin(\theta) & -5\cos^4(\theta)\sin^2(\theta)\end{pmatrix}$ |

monotonically decreasing function in the former case:

$$\mathcal{D}[(\mathcal{F}_1\mathcal{F}_1\cdots\mathcal{F}_1)\rho_0,(\mathcal{F}_1\mathcal{F}_1\cdots\mathcal{F}_1)\rho_1]=|\cos(2\theta)|^n. \quad (17)$$

Unless $2\theta=0,\ \pi,\ 2\pi$, we have $0\leqslant|\cos(2\theta)|<1$; therefore, $|\cos(2\theta)|^n$ converges to zero as $n$ increases, as shown in Fig. 2(c).

*Discerning multiple non-Markovian effects.* Quantum walks have been studied in the presence of various noise models, both Markovian and non-Markovian [21,41]. It is important to mention here that the inferences drawn about the non-Markovian behavior in such cases must take into account the inherent non-Markovian nature of the reduced coin dynamics. To illustrate this point, let us subject the reduced coin state to the RTN channel, $\mathcal{E}:\rho(t)=\mathcal{E}\rho(0)$, described by the following Kraus operators:

$$R_1=\sqrt{\frac{1+\Lambda(t)}{2}}\mathbb{1},\qquad R_2=\sqrt{\frac{1-\Lambda(t)}{2}}\sigma_z. \quad (18)$$

Here,

$$\Lambda(t)=e^{-\gamma t}\left[\cos\left(\gamma t\sqrt{4\frac{a^2}{\gamma^2}-1}\right)+\frac{1}{\sqrt{4\frac{a^2}{\gamma^2}-1}}\sin\left(\gamma t\sqrt{4\frac{a^2}{\gamma^2}-1}\right)\right]. \quad (19)$$

The RTN describes a dephasing noise studied in Ref. [43], with the autocorrelation function, represented by the stochastic variable $\xi$, given by $\langle\xi(t)\xi(s)\rangle=a^2e^{-|t-s|/\tau}$. Here, $a$ signifies the strength of the system-environment coupling, and $\gamma=\frac{1}{2\tau}$ describes the fluctuation rate of the RTN. The channel describes a Markovian (non-Markovian) evolution if $\frac{a^2}{\gamma^2}<0.25$ ($\frac{a^2}{\gamma^2}>0.25$). Next, we define the composition of RTN and QW channels as $[\mathcal{E}\mathcal{F}]_{t=n}$ for $n$ steps, such that

$$\rho_c(t=n)=[\mathcal{E}\mathcal{F}]_{t=n}\rho_c(0)=[\mathcal{E}[\mathcal{F}\rho_c(0)]]_{t=n}, \quad (20)$$

where the map $\mathcal{F}$ is defined in Eq. (12). Figure 4 depicts the behavior of trace distance under this composite map, where RTN is operated both in Markovian and non-Markovian

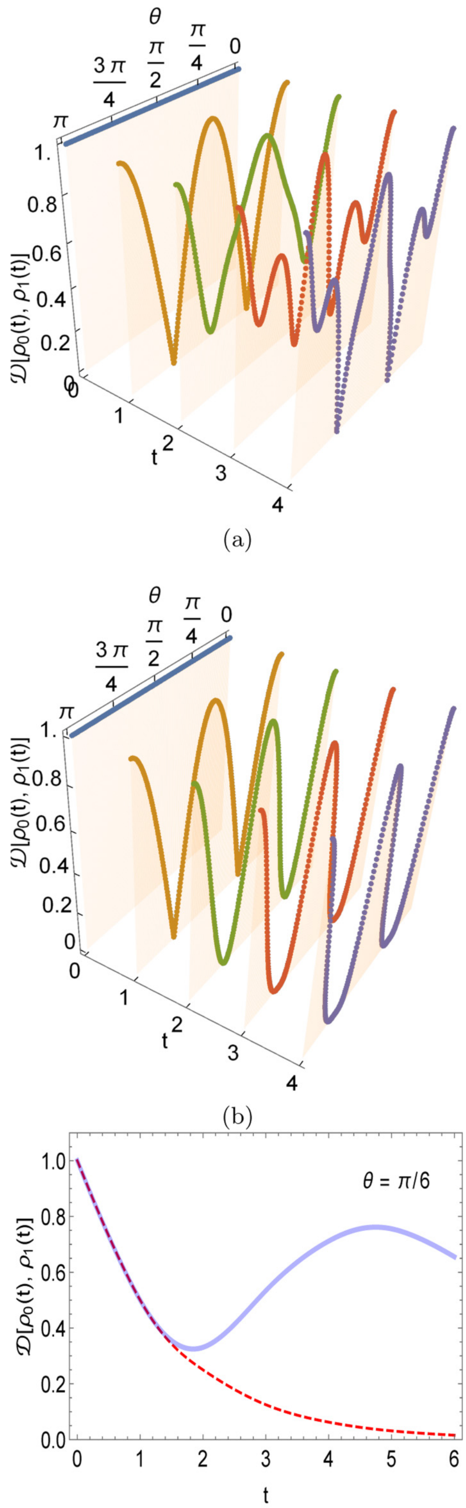


FIG. 2. (a) Trace distance between orthogonal states $|0\rangle$ and $|1\rangle$ subjected to coin dynamics as a function of coin parameter $\theta$ and the number of steps. The $n$th step is realized by applying $\mathcal{F}_n$, defined in Eq. (12). (b) Trace distance between $|0\rangle$ and $|1\rangle$ obtained by subjecting them to an $n$ concatenation of $\mathcal{F}_1$. (c) We compare (a) and (b) for $\theta = \pi/6$, with blue (solid) and red (dashed) curves corresponding to a single $n$-step operation and an $n$-concatenation operation, respectively.

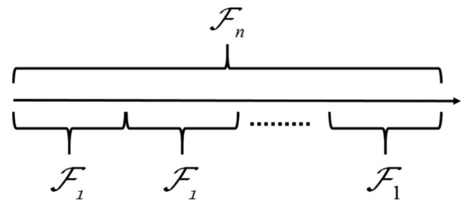


FIG. 3. The $n$-step reduced coin operation obtained in two inequivalent ways. The map $\mathcal{F}_n$ is defined in Eq. (12).

regimes. The nonmonotonic behavior of trace distance in the Markovian regime of the RTN channel is a consequence of the inherent non-Markovian nature of the reduced coin dynamics.

### B. Purity and mixedness under QW channel

The purity of a state quantifies the degree of disorder or mixedness in it. The system-environment interaction is often accompanied with a loss of coherence in the state, leading to mixedness. Thus, purity and mixedness are complementary quantities connected by the following relation [51]:

$$\mathcal{M} = \frac{d}{d-1}(1 - \mathrm{Tr}[\rho^2]). \tag{21}$$

Here, $\mathcal{M}$ is the mixedness and $\mathrm{Tr}[\rho^2]$ is the purity of the $d$-dimensional state $\rho$. Figures 5(a) and 5(b) depict the purity of the output state of the QW channel when the input state is $\cos(\delta/2)\,|0\rangle + \sin(\delta/2)\,|1\rangle$. For both even and odd numbers of steps, the system is found to be in a pure state for $\theta = 0, \pi/2, \pi$. The same quantity is depicted in Figs. 5(c) and 5(d), with respect to the coin parameter $\theta$, for state parameter $\delta = \pi/4$.

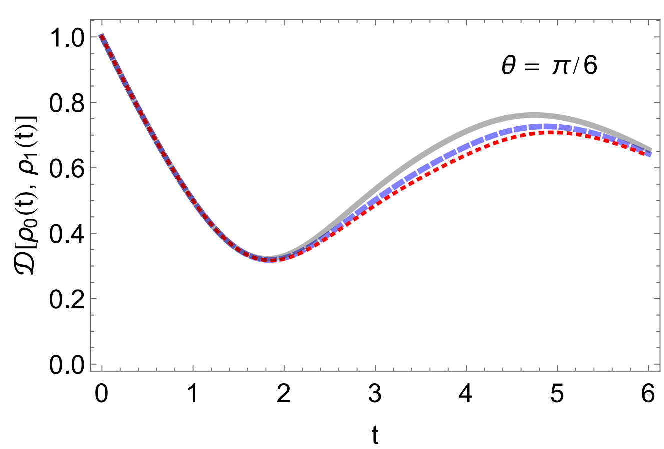


FIG. 4. Trace-distance between states $\mathcal{EF}(|0\rangle\langle 0|)$ and $\mathcal{EF}(|1\rangle\langle 1|)$, where the composite map $\mathcal{EF}$ is defined in Eq. (20). The blue (dashed) and red (dotted) curves correspond to the cases when RTN is operated in Markovian and non-Markovian regimes, respectively. The black curve depict the case in absence of RTN channel. The unexpected nonmonotonous behavior of trace distance in the Markovian regime of RTN is due to the inherent non-Markovian nature of the dynamics.

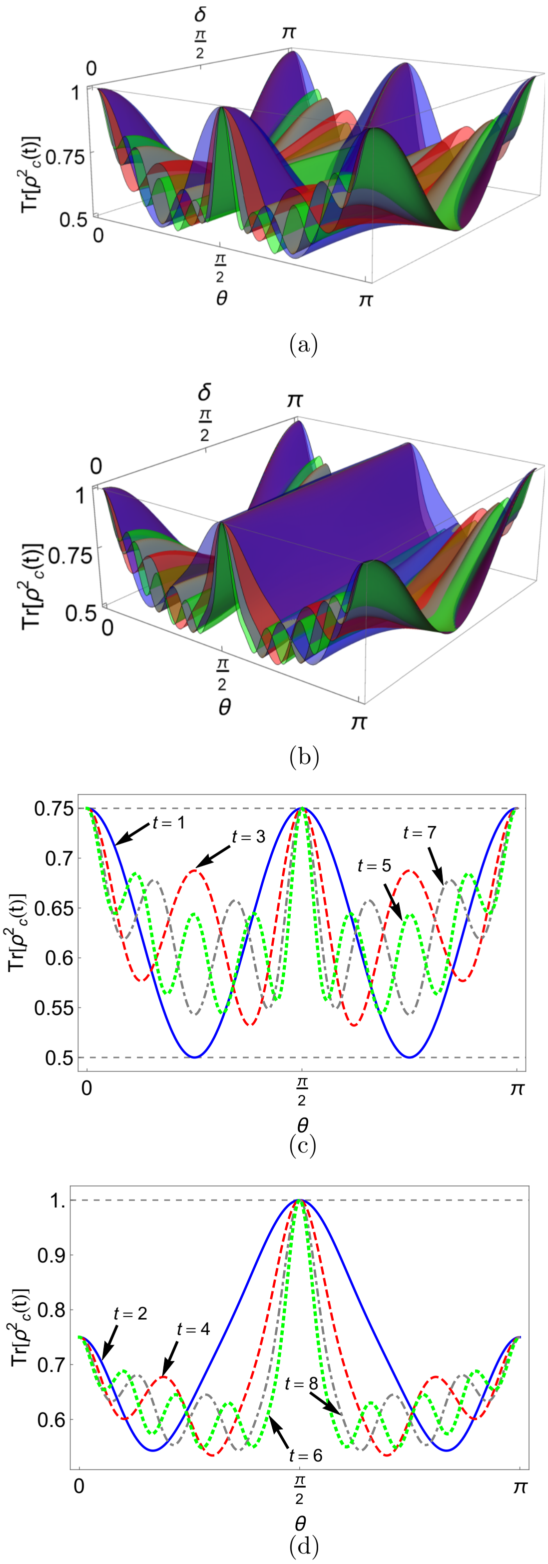


FIG. 5. (a), (b) Depicting the trace of the reduced coin state for a $t$-step QW as a function of the coin parameter $\theta$ and state parameter $\delta$ with with input state $\cos(\delta/2)\,|0\rangle + \sin(\delta/2)\,|1\rangle$. In (a) and (b) the blue, red, gray, and green surfaces correspond to $t = 1, 3, 5, 7$, and $t = 2, 4, 6, 8$, respectively. The same quantity is plotted in (c) and (d) with respect to $\theta$, and $\delta = \pi/4$.

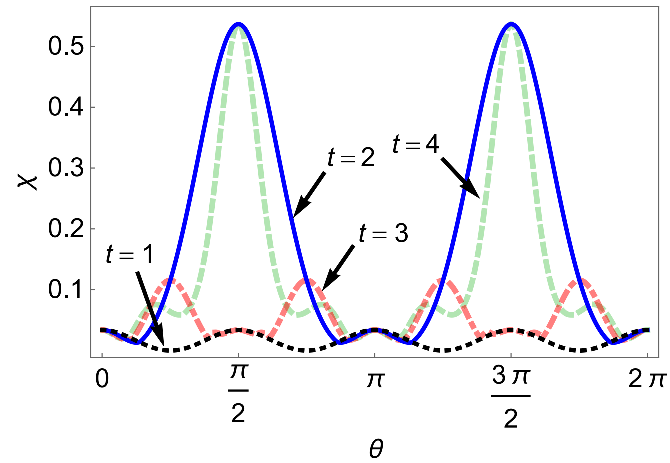


FIG. 6. Maximum of the Holevo quantity $\chi$ as defined in Eq. (22). The input state is taken to be $\rho = p_1\rho_1 + p_2\rho_2$, with $\rho_1 = \frac{1}{4}|0\rangle\langle 0| + \frac{3}{4}|1\rangle\langle 1|$ and $\rho_2 = \frac{1}{6}|+\rangle\langle +| + \frac{5}{3}|-\rangle\langle -|$. The maximization is carried over all $0 \leqslant p_1 < 1$ and $0 \leqslant p_2 < 1$, constrained to $p_1 + p_2 = 1$.

### C. Holevo quantity for QW channel

When a state is subjected to a noise channel, its quantum features get affected, usually manifested in the form of decoherence and dissipation. The amount of information about the input state that can be retrieved from the output state is known as *accessible information*. The accessible information is upper bounded by the Holevo quantity [52] defined as

$$\chi = S\left(\sum_j p_j \mathcal{F}(\rho_j)\right) - \sum_j p_j S(\mathcal{F}(\rho_j)). \tag{22}$$

Here, $\rho_j$ is the set of input states with probability $p_j$, describing the ensemble $\{p_j, \rho_j\}$. The map $\mathcal{F}$ in our case represents the reduced coin dynamics, and is defined in Eq. (12). Let us consider a case when the input state is described by the ensemble $\{p_1\rho_1, p_2\rho_2\}$, with $\rho_1 = \frac{1}{4}|0\rangle\langle 0| + \frac{3}{4}|1\rangle\langle 1|$ and $\rho_2 = \frac{1}{6}|+\rangle\langle +| + \frac{5}{6}|-\rangle\langle -|$. For different numbers of steps, the Holevo quantity, maximized over $0 \leqslant p_1 < 1$ and $0 \leqslant p_2 < 1$, with $p_1 + p_2 = 1$, is depicted in Fig. 6. One infers that the Holevo quantity is suppressed for an odd number of steps.

## IV. CONCLUSION

Recent studies have reported the constructive role of non-Markovian quantum channels over Markovian ones, in enhancing various quantum features of the system. We have characterized the reduced coin dynamics in DTQWs as a non-Markovian quantum channel by analytically computing the Kraus operators for a $t$-step walk. The non-Markovianity is inferred from the P divisibility, reflected by the nonmonotonous behavior of the trace distance between two orthogonal states subjected to the channel. Subtleties arising due to concatenation of a one-step map for $t$ number of steps are highlighted. This could be envisaged to have an impact on the study of memory processes on QW evolutions. The impact of a noisy channel on the purity of a quantum state is studied with respect to the number of steps as well as the channel (coin) parameter. The amount of information about an input state which can be retrieved from the output is bounded by Holevo quantity, and is shown to exhibit different behavior for even and odd numbers of steps. The QW channels, introduced here, add to the important class of non-Markovian channels which help in developing characterization methods for open quantum systems and strategies for various quantum information tasks. The feasibility of experimental implementation of DTQWs in various quantum systems can lead the way towards practical realization of non-Markovian quantum channels presented in this work.

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## APPENDIX: CALCULATION OF $\langle x=\mu|\hat{S}_L^k\hat{S}_R^{t-k}|x=\nu\rangle$

From the definition,

$$\hat{S}_L=\sum_{x=-t}^{t}|x-1\rangle\langle x| \qquad \text{and} \qquad \hat{S}_R=\sum_{x=-t}^{t}|x+1\rangle\langle x|. \tag{A1}$$

Note that $\sum_{x=-t}^{t}|x-1\rangle\langle x|=\sum_{x=-t-1}^{t-1}|x\rangle\langle x+1|$. We propose

$$[\hat{S}_L]^k=\left[\sum_{x=-t-1}^{t-1}|x\rangle\langle x+1|\right]^k=\sum_{x=-t-1}^{t-k}|x\rangle\langle x+k|. \tag{A2}$$

We prove this by induction. The cases with $k=0$ and $k=1$ trivially hold. Let us assume the results holds for $k=p$, so that

$$\begin{aligned}
&\left[\sum_{x=-t-1}^{t-1}|x\rangle\langle x+1|\right]^{p+1}\\
&=\left[\sum_{x=-t-1}^{t-1}|x\rangle\langle x+1|\right]\left[\sum_{x=-t-1}^{t-1}|x\rangle\langle x+1|\right]^{p}\\
&=\left[\sum_{x=-t-1}^{t-1}|x\rangle\langle x+1|\right]\left[\sum_{y=-t-1}^{t-p}|y\rangle\langle y+p|\right]\\
&=\sum_{x=-t-1}^{t-1}\sum_{y=-t-1}^{t-p}|x\rangle\langle x+1|y\rangle\langle y+p|\\
&=\sum_{x=-t-1}^{t-1}\sum_{y=-t-1}^{t-1}|x\rangle\langle y+p|\ \delta_{x+1,y}\\
&=\sum_{x=-t-p}^{t-(p+1)}|x\rangle\langle x+p+1|.
\end{aligned} \tag{A3}$$

The upper limit of $x$ is restricted to $t-(p+1)$, since $y=x+1$; therefore, for $x>t-(p+1)$ we have $y>t-p$, which is greater than the original limit of $y$. Similarly, one can show

$$[\hat{S}_R]^k=\left[\sum_{x=-t}^{t}|x+1\rangle\langle x|\right]^t=\sum_{x=-t}^{t-(k-1)}|x+k\rangle\langle x|. \tag{A4}$$

Using Eqs. (A3) and (A4), we have

$$\begin{aligned}
&\langle x=\mu|\hat{S}_L^k\hat{S}_R^{t-k}|x=\nu\rangle\\
&=\langle x=\mu|\left[\sum_{x=-t-1}^{t-k}|x\rangle\langle x+k|\right]\\
&\times\left[\sum_{y=-t}^{t-(k-1)}|y+t-k\rangle\langle y|\right]|x=\nu\rangle\\
&=\sum_{x=-t-1}^{t-k}\sum_{y=-t}^{t-(k-1)}\delta_{\mu,x}\langle x+k|y+t-k\rangle\delta_{y,\nu}\\
&=\langle\mu+k|\nu+t-k\rangle.
\end{aligned} \tag{A5}$$

Therefore, this quantity is nonzero for $k=(t+\nu-\mu)/2$.


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